

ON GENERALIZED KÄHLER GEOMETRY ON COMPACT LIE GROUPS

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ABSTRACT. We present some fundamental facts about a class of generalized Kähler structures defined by invariant complex structures on compact Lie groups. The main computational tool is the BH-to-GK spectral sequences that relate the bi-Hermitian data to generalized geometry data. The relationship between generalized Hodge decomposition and generalized canonical bundles for generalized Kähler manifolds is also clarified.

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1. INTRODUCTION

Generalized Kähler structures were first introduced by Gualtieri in his thesis [11], as the analogue to classical Kähler structures in the framework of generalized geometry à la Hitchin [16]. Recall that a *generalized complex structure* on a manifold M is an integrable almost complex structure

on $\mathbb{T}M := TM \oplus T^*M$, on which a structure of Courant algebroid can be defined by a closed three form $\gamma \in \Omega^3(M)$. A *generalized Kähler structure* is a pair of generalized complex structures $(\mathbb{J}_+, \mathbb{J}_-)$, satisfying a compatibility condition. In [11] it's shown that generalized Kähler structures are equivalent to *bi-Hermitian structures with torsion* as given by Gates-Hull-Roček [10], where it was argued that they are the most general backgrounds for $N = (2, 2)$ supersymmetry.

Let K be a compact Lie group of dimension $2n$, with Lie algebra \mathfrak{k} . It is well-known [11] that K admits natural generalized Kähler structures. In terms of the corresponding bi-Hermitian structures, they are defined by left and right invariant complex structures on K , which exist by Samelson [29] and Wang [31]. More precisely, let I_+ be a right invariant complex structure and I_- a left invariant one, we have

$$\mathbb{J}_\pm := \frac{1}{2} \begin{pmatrix} I_+ \pm I_- & -(\omega_+^{-1} \mp \omega_-^{-1}) \\ \omega_+ \mp \omega_- & -(I_+^* \pm I_-^*) \end{pmatrix} : \mathbb{T}M \rightarrow \mathbb{T}M$$

where ω_\pm are the Kähler forms of I_\pm given by a bi-invariant metric σ on K . These generalized Kähler structures are in general non-Kähler, while the metric σ is always Gauduchon with respect to both complex structures I_\pm (cf. Lemma 5.3). In particular, degrees are well defined for I_\pm -holomorphic vector bundles, as well as generalized holomorphic vector bundles on K (Hu-Moraru-Seyyedali [19]).

In this note, we present some fundamental facts on a class of generalized Kähler structures on a compact Lie group K . Let $\gamma \in \Omega^3(K)$ be the Cartan 3-form. The corresponding Courant algebroid $\mathbb{T}K$ can be trivialized by the natural extended actions of $K \times K$ (Alekseev-Bursztyn-Meinrenken [1], cf. Meinrenken [26]). Corresponding to the invariant Kähler structures on an even dimensional torus T , we have the notion of *Lie algebraic generalized Kähler structures* on K .

Definition 1.1. *A generalized complex structure on K defined via the Courant trivialization by a complex Lagrangian subalgebra $\mathfrak{L} \subset \mathfrak{d}$ satisfying $\mathfrak{L} \cap (\mathfrak{k} \oplus \mathfrak{k}) = \{0\}$ is called a Lie algebraic generalized complex structure. A generalized Kähler structure $(K, \gamma; \mathbb{J}_\pm, \sigma)$ is a Lie algebraic generalized Kähler structure if both \mathbb{J}_\pm are Lie algebraic generalized complex structures.*

It turns out that such generalized Kähler structures are precisely those defined by the invariant complex structures. We comment that these structures are different from the *invariant structures* considered in the literature, e.g. Alekseevsky-David [2] and the references therein. In general, Lie algebraic generalized Kähler structures are not invariant under the left or right actions of the group; and a Lie algebraic generalized complex structure may not be part of a Lie algebraic generalized Kähler structure (cf. Lemma 5.7).

Many classical notions have their generalized analogues, such as generalized Calabi-Yau manifold [16], generalized holomorphic vector bundles, generalized canonical bundles [11] and generalized Hodge decompositions (Gualtieri [12], Cavalcanti [6]), to name a few. We work out some of these analogues for Lie algebraic generalized Kähler structures. For example, when $\pi_1(K)$ is torsionless, we show (Theorem 5.20) that the degrees of the canonical line bundles are invariants of the group K , (upto the choice of the bi-invariant metric σ).

Theorem 1.2. *Let $\mathfrak{g} = \mathfrak{k} \otimes \mathbb{C}$ be the complexified Lie algebra. With a choice of Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and the corresponding set of positive roots R_+ , let $\rho = \frac{1}{2} \sum_{\alpha \in R_+} h_\alpha \in \mathfrak{h}$. The degree of the canonical line bundle for \mathbb{J}_+ is given by $-2|\rho|^2$, where the norm is induced by the bi-invariant metric on K .*

As a corollary (Corollary 5.22), we recover the result in Cavalcanti [7], that \mathbb{J}_+ is not generalized Calabi-Yau unless K is a torus, in which case, it is part of a generalized Calabi-Yau metric structure (Corollary 5.23).

Let L_+ be the i -eigenbundle of the generalized complex structure \mathbb{J}_+ , which decomposes further into $\pm i$ -eigensubbundles of \mathbb{J}_- :

$$L_+ = \ell_+ \oplus \ell_-$$

then ℓ_{\pm} form a *matched pair* of Lie algebroids (cf. Laurent-Gengoux-Stiénon-Xu [23], Mackenzie [25], Mokri [27] and Lu [24]). The double complex $(\Omega^{p,q}(\overline{L}_+), \bar{\delta}_+, \bar{\delta}_-)$ associated to the matched pair $(\overline{\ell}_+, \overline{\ell}_-)$ appeared first in Gualtieri [15] in describing the deformation theory for \mathbb{J}_{\pm} . The associated spectral sequences are called the *BH-to-GK spectral sequences* (§3). In Lemma 3.5, we show that the differentials $\bar{\delta}_{\pm}$ can be explicitly identified using components of the corresponding Bismut connections.

The *BH-to-GK* spectral sequence for \mathbb{J}_+ -holomorphic vector bundles provides the relation between the generalized Hodge decomposition and cohomology of the generalized canonical bundles. Recall that for a generalized complex structure \mathbb{J} on M with i -eigenbundle L , the *generalized canonical line bundle* $U \subset \wedge^* T_{\mathbb{C}}^* M$ is generated locally by the pure spinors $\chi \in \Omega^*(M)$ defining \overline{L} . It is naturally a \mathbb{J} -holomorphic line bundle [11]. Furthermore, suppose that $\mathbb{J} = \mathbb{J}_+$ is part of a generalized Kähler structure, then $U_{-n,0} := C^{\infty}(U) \subset \Omega^*(M)$ generates, via the spinor actions of ℓ_{\pm} , the *generalized Hodge decomposition* [14] of the twisted de Rham complex $(\Omega^*(M), d_{\gamma} := d + \gamma \wedge)$. In particular, d_{γ} decomposes (cf. (2.8))

$$d_{\gamma} = \delta_+^{\gamma} + \delta_-^{\gamma} + \bar{\delta}_+^{\gamma} + \bar{\delta}_-^{\gamma}$$

For a generalized Kähler manifold, we have (cf. Theorem 3.19)

Theorem 1.3. *The double complex inducing the BH-to-GK spectral sequence is naturally isomorphic to the half of the generalized Hodge decomposition given by $(\bar{\delta}_+^{\gamma}, \bar{\delta}_-^{\gamma})$. More precisely*

$$(\Omega^{p,q}(U; \overline{L}), D_+, D_-) \cong (U_{r,s}, \bar{\delta}_+^{\gamma}, \bar{\delta}_-^{\gamma})$$

where $r = p + q - n$ and $s = p - q$.

When K is semi-simple, it is well-known that the *twisted de Rham cohomology* $H_{\gamma}^*(K)$ vanishes (e.g. Ferreira [9]). We then obtain the following vanishing result (cf. Proposition 5.14).

Theorem 1.4. *For a semi-simple Lie group K with torsionless π_1 , let \mathbb{J} be part of a Lie algebraic generalized Kähler structure and U the generalized canonical line bundle, then $H^*(U; \mathbb{J}) = 0$.*

For general compact Lie groups, the Hodge decomposition can be explicitly described at the level of Lie algebras, via the Courant trivialization (§5.6).

In [15], it is shown that holomorphic reduction induces on $\overline{\ell}_{\pm}$ the structures of I_{\mp} -holomorphic Lie algebroids, which are denoted \mathcal{A}_{\mp} . Using Morita equivalence, [15] further showed that the category of \mathbb{J}_+ -holomorphic vector bundles is equivalent to the categories of (locally free) holomorphic \mathcal{A}_{\pm} -modules (cf. Corollary 3.15, *loc. cit.*). By the explicit identification of the differentials $\bar{\delta}_{\pm}$, we obtain the following refinement (cf. Proposition 3.15).

Theorem 1.5. *On a generalized Kähler manifold, the categories of \mathbb{J}_+ -holomorphic vector bundles and (locally free) \mathcal{A}_{\pm} -modules are isomorphic to each other.*

We note that the above theorem can be obtained as special case from results in [23], while we provide details for completeness.

A key simplification for compact Lie groups comes from the observation that \mathcal{A}_{\mp} are trivial as holomorphic vector bundles (cf. Proposition 5.27). As further applications of the BH-to-GK spectral sequence, we compute the Lie algebroid cohomology $H^*(\overline{L}_+)$ and the \mathbb{J}_+ -Picard group $\text{Pic}_0^+(K)$ of \mathbb{J}_+ -holomorphic line bundles (cf. Corollary 5.30 and Proposition 5.31).

Theorem 1.6. *Suppose that $\mathbb{J} = \mathbb{J}_+$ is part of a generalized Kähler structure on K , and let $L := L_+$ denote the i -eigenbundle of \mathbb{J} . Then*

$$H^*(\overline{L}_+) \cong \wedge^* \mathbb{C}^{2r} \text{ and } \text{Pic}_0^+(K) \cong \text{Pic}_0^+(K) \times \mathbb{C}^r$$

where $2r$ is the rank of K and $\text{Pic}_0^+(K)$ denotes the identity component of the Picard group of I_+ -holomorphic line bundles on K .

The structure of the paper is as follows. We briefly review the general facts about generalized Kähler structures in §2 and recall the Courant trivialization as well as the differential in the Clifford algebra in §4. In §3, we describe the BH-to-GK spectral sequences and the relation to the generalized Hodge decomposition. The more detailed computations for compact Lie groups are contained in the last section §5, where we also identify Lie algebraic generalized Kähler structures.

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2. GENERALIZED KÄHLER GEOMETRY

Let M be a smooth manifold of dimension $2n$, $\gamma \in \Omega^3(M)$ a closed 3-form, and g a Riemannian metric on M . We give a self-contained description of basic facts on generalized Kähler geometry and generalized holomorphic vector bundles. More details can be found in several of Gualtieri's papers [11], [13], [14] and [15].

2.1. Equivalent descriptions. We recall here two equivalent descriptions of generalized Kähler geometry: first as a *bi-Hermitian structure* [10]; then as the analogue to the Kähler structures in the context of generalized geometry [11].

Let I_{\pm} be integrable almost complex structures on M such that g is Hermitian with respect to both of them. Let $\omega_{\pm} = g \circ I_{\pm}$ be the corresponding Kähler form. Then the tuple $(M, \gamma; g, I_{\pm})$ defines a *bi-Hermitian structure* if

$$\pm d^c_{\pm} \omega_{\pm} = \gamma, \text{ where } d^c = i(\bar{\partial} - \partial)$$

For each Hermitian manifold, there is a corresponding *Bismut connection*, for which both the metric and the integrable complex structure are parallel. Here, the corresponding Bismut connections are denoted ∇^{\pm} , then

$$\nabla^{\pm} g = 0 \text{ and } \nabla^{\pm} I_{\pm} = 0$$

The relation to the Levi-Civita connection is given by γ , as will be shown below

$$\nabla_X^{\pm} Y = \nabla_X Y \pm \frac{1}{2} g^{-1} \iota_X \iota_Y \gamma$$

Consider the generalized tangent bundle $\mathbb{T}M = TM \oplus T^*M$, which admits a Dorfman bracket defined by γ :

$$(X + \xi) * (Y + \eta) = [X, Y] + \mathcal{L}_X \eta - \iota_Y d\xi + \iota_X \iota_Y \gamma$$

The natural projection $\mathbb{T}M \rightarrow TM$ is denoted a . A *generalized almost complex structure* $\mathbb{J} : \mathbb{T}M \rightarrow \mathbb{T}M$ satisfies $\mathbb{J}^2 = -\mathbb{I}$ and is orthogonal with respect to the natural pairing

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2} (\iota_X \eta + \iota_Y \xi)$$

Let L denote the i -eigensubbundle of \mathbb{J} in $\mathbb{T}_{\mathbb{C}}M = \mathbb{T}M \otimes_{\mathbb{R}} \mathbb{C}$, then it is maximally isotropic with respect to the pairing \langle, \rangle . The structure \mathbb{J} is *integrable* and is called a *generalized complex structure* when L is *involutive* with respect to the Dorfman bracket. In other words, L is a *Dirac structure*.

A pair of generalized complex structures $(\mathbb{J}_+, \mathbb{J}_-)$ defines a *generalized Kähler structure* (with corresponding metric g) if they commute and such that

$$\mathbb{G} = -\mathbb{J}_+ \mathbb{J}_- = -\mathbb{J}_- \mathbb{J}_+ = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}$$

A generalized Kähler structure induces the following decomposition:

$$\mathbb{T}_{\mathbb{C}}M = C_+^{\mathbb{C}} \oplus C_-^{\mathbb{C}} = L_+ \oplus \bar{L}_+ = L_- \oplus \bar{L}_- = \ell_+ \oplus \ell_- \oplus \bar{\ell}_+ \oplus \bar{\ell}_-$$

where C_\pm are the graphs of $\pm g$ and $C_\pm^\mathbb{C}$ their complexification, L_+ and L_- are respectively the i -eigensubbundles of \mathbb{J}_+ and \mathbb{J}_- , $\ell_+ = L_+ \cap L_-$ and $\ell_- = L_+ \cap \overline{L}_-$. We have $C_\pm^\mathbb{C} = \ell_\pm \oplus \overline{\ell}_\pm$, which defines complex structures I_\pm on TM by the restriction of \mathbb{J}_+ , i.e. for $X \in TM$

$$I_\pm(X) = a(\mathbb{J}_+(X \pm g(X)))$$

Then a gives isomorphisms of ℓ_\pm to the holomorphic tangent bundles $T_{1,0}^\pm M$

$$\ell_\pm = \{X \pm g(X) = X \mp i\iota_X \omega_\pm : X \in T_{1,0}^\pm M\}$$

The condition that ℓ_\pm are involutive is equivalent to

$$(X \mp i\iota_X \omega_\pm) * (Y \mp i\iota_Y \omega_\pm) = [X, Y] \mp i(\mathcal{L}_X \iota_Y \omega_\pm - \iota_Y d\iota_X \omega_\pm) + \iota_X \iota_Y \gamma = [X, Y] \mp i\iota_{[X, Y]} \omega_\pm$$

which implies that

$$\pm i\iota_X \iota_Y d\omega_\pm + \iota_X \iota_Y \gamma = 0 \iff \mp i(dw_\pm)^{2,1}_\pm = \mp i\partial_\pm \omega_\pm = \gamma_\pm^{(2,1)+(3,0)}$$

Since ω_\pm and γ are both real forms, we have

$$\gamma = \mp i(\partial_\pm \omega_\pm - \overline{\partial}_\pm \omega_\pm) = \pm d_\pm^c \omega_\pm$$

This recovers the correspondence ([11, 15]) of generalized Kähler structures with bi-Hermitian structures with torsion.

The Dorfman bracket induces the Bismut connections (Hitchin [17]). Let $X, Y \in TM$, then

$$\nabla_X^\pm Y := a\left\{[(X \mp g(X)) * (Y \pm g(Y))]^\pm\right\} = \nabla_X Y \pm \frac{1}{2}g^{-1}\iota_X \iota_Y \gamma$$

where \bullet^\pm denotes projection to C_\pm respectively. It follows that ∇^\pm preserve g and their torsions are given by $\pm g^{-1}\iota_X \iota_Y \gamma$. Since L_\pm are involutive with respect to the Dorfman bracket, we see that ℓ_\pm as well as their conjugates are also involutive. In particular

$$Y \in T_{1,0}^+ M \implies \nabla_X^+ Y \in a\{((\ell_- \oplus \overline{\ell}_-) * \ell_+)^+\} \subseteq a\{(\ell_- \oplus \ell_+ \oplus \overline{\ell}_-)^+\} = a(\ell_+) = T_{1,0}^+ M$$

It follows that ∇^+ preserves I_+ . Similarly, ∇^- preserves I_- . We will use ∇^\pm to denote the induced Bismut connection on T^*M :

$$X\alpha(Y) = (\nabla_X^\pm \alpha)(Y) + \alpha(\nabla_X^\pm Y) \implies \nabla^\pm \alpha = \nabla \alpha \pm \frac{1}{2}\iota_{\alpha^\flat} \gamma$$

Since ∇^\pm preserve I_\pm respectively, they preserve $T_{\pm}^{0,1} M$ respectively as well.

2.2. Holomorphic reduction. Consider the decomposition of the Courant algebroid $\mathbb{T}_\mathbb{C}M$ defined by a generalized Kähler structure:

$$(2.1) \quad \mathbb{T}_\mathbb{C}M = \ell_+ \oplus \ell_- \oplus \overline{\ell}_+ \oplus \overline{\ell}_-$$

Reduction with respect to $\overline{\ell}_\pm$ defines I_\pm -holomorphic Courant algebroids

$$0 \rightarrow T_{\pm}^{1,0} M \rightarrow \mathcal{E}_\pm \rightarrow T_{1,0}^\pm M \rightarrow 0$$

For example, the I_+ -holomorphic vector bundle \mathcal{E}_+ is given by

$$\mathcal{E}_+ := \frac{(\overline{\ell}_+)^{\perp}}{\overline{\ell}_+} = \frac{\overline{\ell}_+ \oplus \overline{\ell}_- \oplus \ell_-}{\overline{\ell}_+} \cong \overline{\ell}_- \oplus \ell_-$$

where the I_+ -holomorphic structure is induced by the Dorfman bracket

$$(2.2) \quad \overline{\partial}_{+,X} \mathfrak{Y}_A = (\mathfrak{X} * \mathfrak{Y})_A \text{ for } \mathfrak{X} \in \overline{\ell}_+, X = a(\mathfrak{X}) \in T_{0,1}^+ M, \mathfrak{Y} \in (\overline{\ell}_+)^{\perp}$$

and \bullet_A denotes the equivalence class. Under this reduction, the Lie algebroid \overline{L}_+ induces an I_+ -holomorphic Lie subalgebroid in \mathcal{E}_+ :

$$\overline{L}_+ = \overline{\ell}_+ \oplus \overline{\ell}_- \implies \mathcal{A}_+ := \frac{\overline{\ell}_+ \oplus \overline{\ell}_-}{\overline{\ell}_+} \cong \overline{\ell}_- \subset \mathcal{E}_+$$

Similarly, reduction by $\bar{\ell}_-$ defines the I_- -holomorphic vector bundle $\mathcal{E}_- \cong \bar{\ell}_+ \oplus \ell_+$ and $\bar{\mathcal{L}}_+$ induces an I_- -holomorphic Lie subalgebroid $\mathcal{A}_- \cong \bar{\ell}_+$ in \mathcal{E}_- . We caution here that the isomorphisms in this paragraph are only isomorphisms of complex vector bundles.

We consider the I_+ -holomorphic Lie algebroid \mathcal{A}_+ . For $\mathfrak{X}_\pm \in C^\infty(\bar{\ell}_\pm)$, write $\mathfrak{X} = \mathfrak{X}_+ + \mathfrak{X}_-$. By definition, we have $\mathfrak{X}_A = (\mathfrak{X}_-)_A$. The bracket *does not* descent to $C^\infty(\mathcal{A}_+)$, because

$$[\mathfrak{X}_A, \mathfrak{Y}_A] = (\mathfrak{X} * \mathfrak{Y})_A = (\mathfrak{X}_- * \mathfrak{Y}_- + \mathfrak{X}_+ * \mathfrak{Y}_- + \mathfrak{X}_- * \mathfrak{Y}_+)_A$$

becomes contradictory, for example, when $\mathfrak{X}_- = \mathfrak{Y}_+ = 0$. On the other hand, when restricted to the I_+ -holomorphic sections, we have

$$[\mathfrak{X}_A, \mathfrak{Y}_A] = (\mathfrak{X}_- * \mathfrak{Y}_-)_A = ([\mathfrak{X}_-, \mathfrak{Y}_-])_A$$

Thus, $\bar{\ell}_\mp$ are the *smooth* Lie algebroids underlying \mathcal{A}_\pm .

2.3. Lie algebroid modules. Let L be a smooth Lie algebroid. The *Lie algebroid de Rham complex* $(\Omega^\bullet(L), d_L)$ is

$$0 \rightarrow C^\infty(M) \xrightarrow{d_L} C^\infty(L^*) \xrightarrow{d_L} C^\infty(\wedge^2 L^*) \xrightarrow{d_L} \dots$$

where it is customary to denote $\Omega^k(L) = C^\infty(\wedge^k L^*)$. The differential d_L is defined in the standard fashion. For $\alpha \in \Omega^k(L)$, and $s_0, \dots, s_k \in C^\infty(L)$ with $X_i = a(s_i)$,

$$(2.3) \quad \begin{aligned} d_L \alpha(s_0, \dots, s_k) &:= \sum_{j=0}^k (-1)^j X_j \alpha(s_0, \dots, \hat{s}_j, \dots, s_k) \\ &\quad + \sum_{i < j} (-1)^{i+j} \alpha(s_i * s_j, s_0, \dots, \hat{s}_i, \dots, \hat{s}_j, \dots, s_k) \end{aligned}$$

The homology of this complex is the *Lie algebroid cohomology*, denoted $H^k(L)$.

Let V be a vector bundle, an L -connection is a derivation $D : V \rightarrow V \otimes L^*$:

$$D(fv) = d_L f \otimes v + f Dv \text{ for } f \in C^\infty(M)$$

It is naturally extended to $\Omega^k(L; V) := C^\infty(V \otimes \wedge^k L^*)$ using d_L :

$$(2.4) \quad D(\alpha \otimes v) = d_L \alpha \otimes v + (-1)^k \alpha \otimes Dv$$

We say that (V, D) (or simply V) is an L -module if $D \circ D = 0$, i.e. D is flat. In this case, there is an induced complex $(\Omega^*(V; L), D)$, whose homology is denoted $H^*(V; L)$.

For a holomorphic Lie algebroid \mathcal{L} , we have correspondingly the *holomorphic Lie algebroid de Rham complex* $(\wedge^* \mathcal{L}, \partial_{\mathcal{L}})$. For local holomorphic sections $\alpha \in \wedge^k \mathcal{L}^*$ and $s_0, \dots, s_k \in \mathcal{L}$, with $X_i = a(s_i)$ local holomorphic vector fields, the righthand side of (2.3) defines $\partial_{\mathcal{L}} \alpha(s_0, \dots, s_k)$. The hyperhomology is denoted $\mathbb{H}^*(\mathcal{L})$. Similarly, we can define the notion of \mathcal{L} -connection on a holomorphic vector bundle V , as well as the notion of \mathcal{L} -modules¹. For an \mathcal{L} -module V , the hyperhomology of the induced complex $(V \otimes \wedge^* \mathcal{L}^*, \partial_{\mathcal{L}})$ is denoted $\mathbb{H}^*(V; \mathcal{L})$.

Remark 2.1. For a holomorphic vector bundle V , the classical Atiyah class $\alpha(V) \in H^1(T_{1,0}^* M \otimes \text{End}(V))$ is the obstruction of existence of a holomorphic connection (Atiyah [4]). One can show that the existence of an \mathcal{L} -connection is obstructed by the corresponding Atiyah class $a^* \alpha(V) \in H^1(\mathcal{L}^* \otimes \text{End}(V))$. When V is a line bundle, $a^* \alpha(V) \in H^1(\mathcal{L}^*)$. For an \mathcal{L} -module V , we have

$$a^* \alpha(V) = 0$$

We note that in general, this does not imply that $\alpha(V) = 0$.

¹A more general notion is a *sheaf of \mathcal{L} -modules* (cf. Tortella [30]). The \mathcal{L} -modules considered here are the locally free ones.

Definition 2.2. Let \mathcal{L} be a holomorphic Lie algebroid. Two \mathcal{L} -modules V_1 and V_2 are equivalent if there exists holomorphic bundle isomorphism $f : V_1 \rightarrow V_2$ covering identity that preserves the \mathcal{L} -connections. The \mathcal{L} -Picard group $\text{Pic}^{\mathcal{L}}(M)$ consists of the equivalence classes of \mathcal{L} -modules of rank 1, where the group structure is given by tensor product.

The group $\text{Pic}^{\mathcal{L}}(M)$ is obviously abelian.

Lemma 2.3. The map $\text{Pic}^{\mathcal{L}}(M) \rightarrow \text{Pic}(M)$ forgetting the \mathcal{L} -connection is a group homomorphism, whose kernel $\text{Pic}_0^{\mathcal{L}}(M)$ is the subgroup where the underlying holomorphic line bundle is trivial. \square

2.4. \mathbb{J}_+ -holomorphic vector bundles. Let \mathbb{J} be a generalized complex structure, with the i -eigensubbundle L . A \mathbb{J} -holomorphic vector bundle $V \rightarrow M$ is an \overline{L} -module, i.e. a complex vector bundle with a flat \overline{L} -connection:

$$D : C^\infty(V) \rightarrow C^\infty(V \otimes \overline{L}^*), D(fv) = d_{\overline{L}}f \otimes v + fDv$$

such that $D \circ D = 0$. The homology of the associated complex $(\Omega^*(V; \overline{L}), D)$ is denoted $H^*(V; \mathbb{J})$. We say that a section $v \in \Omega^0(V)$ is \mathbb{J} -holomorphic if $Dv = 0$. Similar to the classical case, $H^0(V; \mathbb{J})$ consists of global holomorphic sections of V .

Suppose now that $\mathbb{J} = \mathbb{J}_+$ is part of a generalized Kähler structure. Since $\overline{L} = \overline{\ell}_+ \oplus \overline{\ell}_-$, D naturally decomposes as $D = D_+ + D_-$:

$$D_\pm : C^\infty(V) \rightarrow C^\infty(V \otimes \overline{\ell}_\pm^*), D_\pm(fv) = d_{\overline{\ell}_\pm}f \otimes v + fD_\pm v$$

which implies that D_\pm are $\overline{\ell}_\pm$ -connections on V . We use the same notations D_\pm to denote their extensions to $\Omega^*(V; \overline{\ell}_\pm)$ by (2.4).

Proposition 2.4. Let $\mathbb{J} = \mathbb{J}_+$ be part of a generalized Kähler structure. A \mathbb{J} -holomorphic vector bundle is an I_\pm -holomorphic vector bundle on (M, I_+, I_-) . A section $v \in \Omega^0(V)$ is \mathbb{J} -holomorphic iff it is a holomorphic section in both I_\pm -holomorphic structures.

Proof: Let $s_\pm \in \overline{\ell}_\pm$ respectively and write $s = s_+ + s_- \in \overline{L}$, then for $v \in V$, we have

$$D_{\pm, s_\pm}(v) = D_{s_\pm}(v)$$

The flatness of D is equivalent to the following identity for any $s, t \in \overline{L}$ and $v \in V$:

$$(2.5) \quad D_s D_t(v) - D_t D_s(v) - D_{s*t}(v) = 0$$

By setting $s_\bullet = t_\bullet = 0$, $\bullet = \pm$, in the above equality we see that D_\pm are flat as well. Since $\overline{\ell}_\pm \cong T_{0,1,\pm}M$ as Lie algebroids, we see that D_\pm defines I_\pm -holomorphic structures on V . The last statement is straightforward. \square

In [15], using the notion of Morita equivalence, it is shown that the category of \mathbb{J}_+ -holomorphic vector bundles is equivalent to the category of \mathcal{A}_\bullet -modules, where $\bullet = +$ or $-$. We will show that this equivalence is in fact an isomorphism of categories (Proposition 3.15).

2.5. Canonical line bundle. Let \mathbb{J} be a generalized complex structure. The \mathbb{J} -canonical bundle U is the pure spinor line bundle defining the Dirac structure \overline{L} , i.e.

$$\overline{L} = \{\mathfrak{X} | \mathfrak{X} \cdot C^\infty(U) = 0\}$$

Let $\chi \in C^\infty(U)$ and $d_\gamma := d + \gamma \wedge$ the twisted de Rham differential, it's shown in [11] that the integrability of \mathbb{J} is equivalent to $d_\gamma \chi \in C^\infty(L) \cdot C^\infty(U)$. Since the spinor action defines an isomorphism of $L \otimes U$ onto the image, d_γ defines an \overline{L} -connection under the identification $L \cong \overline{L}^*$ by $2\langle \cdot, \cdot \rangle$:

$$(2.6) \quad D_U : C^\infty(U) \rightarrow C^\infty(U \otimes \overline{L}^*) : D_{U,s}\chi := s \cdot d_\gamma \chi \text{ for } s \in C^\infty(\overline{L})$$

It turns out that D_U is flat ([11]) and U is naturally a \mathbb{J} -holomorphic line bundle.

When $\mathbb{J} = \mathbb{J}_+$ is part of a generalized Kähler structure, the canonical line bundle generates the *generalized Hodge decomposition* of the twisted de Rham complex $(\Omega^*(M), d_\gamma)$ via the spinor actions of \mathbb{J}_\pm ([12] and Baraglia [5]). More precisely, let $U_{-n,0} := C^\infty(U) \subset \Omega^*(M)$, then

$$U_{r,s} := (C^\infty(\wedge^p \ell_+) \otimes C^\infty(\wedge^q \ell_-)) \cdot U_{-n,0} \subset \Omega^*(M)$$

with $r = p + q - n$ and $s = p - q$, by the spinor action of $C^\infty(\ell_\pm)$ on $U_{-n,0}$. The differential d_γ decomposes as

$$(2.7) \quad d_\gamma = \partial_+^\gamma + \bar{\partial}_+^\gamma \text{ with } \partial_+^\gamma := \delta_+^\gamma + \delta_-^\gamma \text{ and } \bar{\partial}_+^\gamma := \bar{\delta}_+^\gamma + \bar{\delta}_-^\gamma$$

where $\bar{\delta}_\pm^\gamma : U_{r,s} \rightarrow U_{r+1,s\pm 1}$ and $\delta_\pm^\gamma : U_{r,s} \rightarrow U_{r-1,s\mp 1}$. In terms of the gradings given by p and q , we have

$$(2.8) \quad \begin{aligned} \bar{\delta}_+^\gamma &: (C^\infty(\wedge^p \ell_+) \otimes C^\infty(\wedge^q \ell_-)) \cdot U_{-n,0} \rightarrow (C^\infty(\wedge^{p+1} \ell_+) \otimes C^\infty(\wedge^q \ell_-)) \cdot U_{-n,0} \\ \bar{\delta}_-^\gamma &: (C^\infty(\wedge^p \ell_+) \otimes C^\infty(\wedge^q \ell_-)) \cdot U_{-n,0} \rightarrow (C^\infty(\wedge^p \ell_+) \otimes C^\infty(\wedge^{q+1} \ell_-)) \cdot U_{-n,0} \\ \delta_+^\gamma &: (C^\infty(\wedge^p \ell_+) \otimes C^\infty(\wedge^q \ell_-)) \cdot U_{-n,0} \rightarrow (C^\infty(\wedge^{p-1} \ell_+) \otimes C^\infty(\wedge^q \ell_-)) \cdot U_{-n,0} \\ \delta_-^\gamma &: (C^\infty(\wedge^p \ell_+) \otimes C^\infty(\wedge^q \ell_-)) \cdot U_{-n,0} \rightarrow (C^\infty(\wedge^p \ell_+) \otimes C^\infty(\wedge^{q-1} \ell_-)) \cdot U_{-n,0} \end{aligned}$$

In [12], it's shown that the Laplacians of the various differentials coincide up to scalars, which gives the generalized Hodge decomposition of the twisted de Rham cohomology $H_\gamma^*(M)$.

3. A SPECTRAL SEQUENCE

We describe in this section a computational tool, the *BH-to-GK* spectral sequence, relating the cohomology groups in the bi-Hermitian data to those in the generalized Kähler data.

3.1. The double complex. Let's consider the decomposition $L := L_+ = \ell_+ \oplus \ell_-$. This is *not* a direct sum as Lie algebroids. Although the brackets on L as well as the summands are given by the restriction of the Dorfman bracket on $\mathbb{T}M$, in general

$$\ell_+ * \ell_- \neq 0$$

Analysing this non-vanishing leads to the spectral sequence.

We have the bundle isomorphism

$$\wedge^* \bar{L}^* \cong \bigoplus_{p,q} \wedge^p \bar{\ell}_+^* \otimes \wedge^q \bar{\ell}_-^* \cong \bigoplus_{p,q} \wedge^p T_+^{0,1} M \otimes \wedge^q T_-^{0,1} M$$

which induces the isomorphism as linear spaces

$$\Omega^*(\bar{L}) \cong \bigoplus_{p,q} \Omega^{p,q}(\bar{L}) \text{ where } \Omega^{p,q}(\bar{L}) = \Omega^p(\bar{\ell}_+) \otimes \Omega^q(\bar{\ell}_-) \cong \Omega_+^{0,p}(M) \otimes \Omega_-^{0,q}(M)$$

Let $\alpha \in \Omega_+^{0,p}(M)$ and $\beta \in \Omega_-^{0,q}(M)$. The same notations are used for the corresponding elements in $\Omega^*(\bar{\ell}_\pm)$. Then $\alpha \wedge \beta \in \Omega^{p,q}(\bar{L})$. By definition, we have

$$d_{\bar{L}}(\alpha \wedge \beta) = d_{\bar{L}}\alpha \wedge \beta + (-1)^p \alpha \wedge d_{\bar{L}}\beta$$

Lemma 3.1. *Let $\bar{\partial}^\mp : \Omega^*(M) \xrightarrow{\nabla^\pm} \Omega^*(M) \otimes \Omega^1(M) \rightarrow \Omega^*(M) \otimes \Omega_\mp^{0,1}(M)$. Then*

$$d_{\bar{L}}|_{\Omega^{p,0}(\bar{L})} = \bar{\partial}_+ + \bar{\partial}^- \text{ while } d_{\bar{L}}|_{\Omega^{0,q}(\bar{L})} = \bar{\partial}_- + \bar{\partial}^+$$

Proof: We prove the $+$ -case, by directly comparing the two differentials $d_{\bar{L}}$ and $d_{\bar{\ell}_+}$. Consider the sections $s_i \in \bar{\ell}_+$ for $0 < i \leq p$, $r_\pm \in \bar{\ell}_\pm$ and $r = r_+ + r_- \in L$. Suppose that $X_i = a(s_i)$,

$Z_{\pm} = a(r_{\pm})$ and $Z = a(r)$. We compute for $\alpha \in \Omega_+^{0,p}(M)$

$$\begin{aligned} (d_{\overline{L}}\alpha)(r, s_1, \dots, s_p) &= Z\alpha(s_1, \dots, s_p) + \sum_{i=1}^p (-1)^i X_i \alpha(r_+, \dots, \hat{s}_i, \dots, s_p) \\ &+ \sum_{i=1}^p (-1)^i \alpha((r * s_i)^+, \dots, \hat{s}_i, \dots, s_p) + \sum_{i < j} (-1)^{i+j} \alpha(s_i * s_j, r_+, \dots, \hat{s}_i, \dots, \hat{s}_j, \dots, s_p) \\ &= (d_{\overline{L}_+}\alpha)(r_+, s_1, \dots, s_p) + Z_- \alpha(s_1, \dots, s_p) + \sum_{i=1}^p (-1)^i \alpha((r_- * s_i)^+, \dots, \hat{s}_i, \dots, s_p) \end{aligned}$$

When we apply the identification of ℓ_{\pm} with the holomorphic tangent bundles with respect to I_{\pm} , we see that the first term corresponds to $\overline{\partial}_+ \alpha$ and last two terms give precisely $\overline{\partial}_{r_-}^- \alpha$. \square

We are ready to describe the BH-to-GK spectral sequence. By Lemma 3.1, on $\Omega^{p,q}(\overline{L})$, we define

$$(3.1) \quad \overline{\delta}_+ := \overline{\partial}_+ + \overline{\partial}^+ : \Omega^{p,q}(\overline{L}) \rightarrow \Omega^{p+1,q}(\overline{L}) \text{ and } \overline{\delta}_- := \overline{\partial}_- + \overline{\partial}^- : \Omega^{p,q}(\overline{L}) \rightarrow \Omega^{p,q+1}(\overline{L})$$

Then $d_{\overline{L}} = \overline{\delta}_+ + \overline{\delta}_-$. For example, with $\alpha \in \Omega^{p,0}(\overline{L})$ and $\beta \in \Omega^{0,q}(\overline{L})$,

$$\overline{\delta}_+(\alpha \wedge \beta) = \overline{\partial}_+ \alpha \wedge \beta + (-1)^p \alpha \wedge \overline{\partial}^+ \beta$$

In particular, for $\alpha \in \Omega^{p,0}(\overline{L}) \cong \Omega_+^{0,p}(M)$, we have

$$\overline{\delta}_+ \alpha = \overline{\partial}_+ \alpha \text{ and } \overline{\delta}_- \alpha = \overline{\partial}^- \alpha$$

and similarly for $\beta \in \Omega^{0,q}(\overline{L}) \cong \Omega_-^{0,q}(M)$, we have

$$\overline{\delta}_+ \beta = \overline{\partial}^+ \beta \text{ and } \overline{\delta}_- \beta = \overline{\partial}_- \beta$$

Proposition 3.2. *The spectral sequences associated with the double complex $(\Omega^{p,q}(\overline{L}), \overline{\delta}_+, \overline{\delta}_-)$ compute the Lie algebroid cohomology $H^*(\overline{L})$.*

Proof: From $d_{\overline{L}}^2 = 0$, we get $\overline{\delta}_+^2 = 0$, $[\overline{\delta}_+, \overline{\delta}_-] = 0$ and $\overline{\delta}_-^2 = 0$. Thus, $(\Omega^{p,q}(\overline{L}), \overline{\delta}_+, \overline{\delta}_-)$ is indeed a double complex. By definition, the corresponding spectral sequences compute $H^*(\overline{L})$. \square

Remark 3.3. The double complex $(\Omega^{p,q}(\overline{L}), \overline{\delta}_+, \overline{\delta}_-)$ appeared already in the proof of Proposition 3.11 in [15]. Here we identify explicitly the differentials using the bi-Hermitian data.

Definition 3.4. *The spectral sequences associated to the double complex $(\Omega^{p,q}(\overline{L}), \overline{\delta}_+, \overline{\delta}_-)$ are the BH-to-GK spectral sequences for the generalized Kähler manifold $(M, \gamma; \mathbb{J}_{\pm}, \sigma)$.*

The BH-to-GK spectral sequences are related to the holomorphic reduction ([15], and §2.2).

Lemma 3.5. *The operators $\overline{\delta}_{\pm}$ on $\overline{\ell}_{\mp}^*$ defined in (3.1) is induced by the operators $\overline{\partial}_{\pm}$ on \mathcal{A}_{\pm} defined in (2.2) under the natural isomorphism $\overline{\ell}_{\mp} \cong \mathcal{A}_{\pm}$.*

Proof: This follows essentially by comparing the definitions in (3.1) and (2.2). We write down the proof for \mathcal{A}_+ . Let $\alpha_A \in \mathcal{A}_+^*$, with α the corresponding element in $\Omega^{0,1}(\overline{L}) = \overline{\ell}_-^*$ under the natural isomorphism. Let $\mathfrak{X}_+ \in \overline{\ell}_+$ and $\mathfrak{Y}_- \in \overline{\ell}_-$, with $X = a(X_+) \in T_{0,1}^+ M$ and $(\mathfrak{Y}_-)_A \in \mathcal{A}_+$. We compute

$$\begin{aligned} (\overline{\partial}_{+,X} \alpha_A)((\mathfrak{Y}_-)_A) &= X \alpha_A((\mathfrak{Y}_-)_A) - \alpha_A(\overline{\delta}_{+,X}(\mathfrak{Y}_-)_A) \\ &\mapsto X \alpha(\mathfrak{Y}_-) - \alpha((\mathfrak{X}_+ * \mathfrak{Y}_-)^-) = (\overline{\partial}_{+,X} \alpha)(\mathfrak{Y}_-) \end{aligned}$$

Thus $\overline{\partial}_+$ and $\overline{\delta}_+$ coincide on $\mathcal{A}_+^* \cong \overline{\ell}_-^*$. \square

We can now say a little more about the spectral sequences (cf. [15]). Look vertically, Lemma 3.1 implies that the double complex $(\Omega^{p,q}(\overline{L}), \overline{\delta}_+, \overline{\delta}_-)$ gives a resolution of the holomorphic de Rham

complex of \mathcal{A}_- . The first page of the *first* spectral sequence $({}_I E_r^{p,q}, d_r)$ consists of $\bar{\partial}_-$ -Dolbeault cohomologies:

$${}_I E_1^{p,q} = H_{\bar{\partial}_-}^{0,q}(\wedge^p \mathcal{A}_-^*) \text{ with } d_1 \text{ induced by } \bar{\partial}_+$$

Similarly, the first page of the *second* spectral sequence is

$${}_{II} E_1^{p,q} = H_{\bar{\partial}_+}^{0,q}(\wedge^p \mathcal{A}_+^*) \text{ with } d'_1 \text{ induced by } \bar{\partial}_-$$

Proposition 3.6. $\mathbb{H}^*(\mathcal{A}_+) \cong \mathbb{H}^*(\mathcal{A}_-) \cong H^*(\bar{L}_+)$. □

Remark 3.7. The holomorphic Lie algebroids \mathcal{A}_\pm form a *matched pair* (cf. [15], Laurent-Gengoux-Stiénon-Xu [23], Mackenzie [25], Mokri [27] and Lu [24]). Proposition 3.6 corresponds to, for example, Theorem 4.11 of [23].

Example 3.8. Let (M, J, ω) be a Kähler manifold, which can be seen as a generalized Kähler manifold as following:

$$\mathbb{J}_+ = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix} \text{ and } \mathbb{J}_- = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

It follows that $\bar{L}_+ = T_{0,1}M \oplus T^{1,0}M$, and the decomposition into $\bar{\ell}_\pm$ is given by the metric g :

$$\bar{\ell}_\pm = \{X \pm g(X) : X \in T_{0,1}M\}$$

Thus, $I_+ = I_- = J$. The holomorphic reduction in this case endows, for example on $\bar{\ell}_-$, the holomorphic structure given by

$$\bar{\partial}_X(Y - g(Y)) \equiv (X + g(X)) * (Y - g(Y)) \equiv \nabla_X Y - g(\nabla_X Y) \pmod{\bar{\ell}_+} \text{ for } X, Y \in C^\infty(T_{0,1}M)$$

where ∇ is the Levi-Civita connection. By the Kähler condition, ∇ preserves J , which implies that $\nabla_X Y \in C^\infty(T_{0,1}M)$ as well. Let $\nabla^{0,1}$ denote the $(0,1)$ -component of ∇ , then we showed that

$$\bar{\partial} = \nabla^{0,1} \text{ on } C^\infty(\bar{\ell}_-) \cong C^\infty(T_{0,1}M)$$

Lemma 3.5 implies that $\nabla^{0,1}$ induces the operator $\bar{\partial}_+$ in the corresponding double complex $\Omega^{p,q}(\bar{L})$. The metric g defines an isomorphism of vector bundles:

$$\bar{\ell}_- \rightarrow T_{0,1}M \xrightarrow{g} T^{1,0}M \Rightarrow \bar{\ell}_-^* \cong T_{1,0}M$$

The induced holomorphic structure on $T_{1,0}M$ coincides with the standard holomorphic structure as the holomorphic tangent bundle. By Proposition 3.6, we get $H^*(\bar{L}_+) \cong \mathbb{H}^*(T^{1,0}M)$. In particular, the first page of the spectral sequence is given by the Dolbeault cohomology of the sheaves of the holomorphic multivectors:

$$E_1^{p,q} = H_{\bar{\partial}}^{0,q}(\wedge^p T_{1,0}M)$$

Since in this case, $I_+ = I_-$, ℓ_+ can be used in the above arguments and arrive at the same conclusion.

3.2. \mathbb{J}_+ -holomorphic vector bundles. The *BH-to-GK* spectral sequence induces a corresponding spectral sequence for \mathbb{J}_+ -holomorphic vector bundles. Again, we write $L := L_+$.

Let (V, D) be a \mathbb{J} -holomorphic vector bundle. Then we have the decomposition $D = D_+ + D_-$ (see §2.4) into $\bar{\ell}_\pm$ -connections. The extension of the decomposition to $\Omega^*(V; \bar{L})$ is induced by the corresponding decompositions $\Omega^*(\bar{L}) = \bigoplus_{p,q} \Omega^{p,q}(\bar{L})$ and $d_{\bar{L}} = \bar{\partial}_+ + \bar{\partial}_-$:

$$\Omega^*(V; \bar{L}) = \bigoplus_{p,q} \Omega^{p,q}(V; \bar{L}) \text{ where } \Omega^{p,q}(V; \bar{L}) = C^\infty(V \otimes \wedge^p \bar{\ell}_+^* \otimes \wedge^q \bar{\ell}_-^*)$$

We write down the extension of D_+ :

$$(3.2) \quad D_+ : \Omega^{p,q}(V; \bar{L}) \rightarrow \Omega^{p+1,q}(V; \bar{L}) : D_+(\alpha \otimes v) = \bar{\partial}_+ \alpha \otimes v + (-1)^{p+q} \alpha \otimes D_+ v$$

Since $D^2 = 0$, it is straightforward to verify that $(\Omega^{p,q}(V; \bar{L}), D_+, D_-)$ is a double complex. The equation $[D_+, D_-] = 0$ is called the *commutation relation*.

Definition 3.9. *The spectral sequences associated to the double complex $(\Omega^{p,q}(V; \bar{L}), D_+, D_-)$ is the BH-to-GK spectral sequences for the \mathbb{J}_+ -holomorphic vector bundle V .*

The commutation relation helps us understand the equivalence of categories in [15]. We start with the following improvement on Proposition 2.4.

Lemma 3.10. *A complex vector bundle V is \mathbb{J}_+ -holomorphic iff it is an I_\pm -holomorphic vector bundle and the commutation relation holds for the induced differentials D_\pm .*

Proof: We show the \Leftarrow direction. Let $\bar{\partial}_\pm$ denote the I_\pm -holomorphic structures on V , i.e.

$$\bar{\partial}_\pm : C^\infty(V) \rightarrow C^\infty(V \otimes T_{\pm}^{0,1}M) \text{ and } \bar{\partial}_\pm \circ \bar{\partial}_\pm = 0$$

Composing with the isomorphisms $T_{\pm}^{0,1}M \cong \bar{\ell}_\pm^*$ gives the operators $D_\pm : C^\infty(V) \rightarrow C^\infty(V \otimes \bar{\ell}_\pm^*)$. Let $D = D_+ + D_-$. Since $d_{\bar{L}} = d_{\bar{\ell}_+} + d_{\bar{\ell}_-}$ on functions, we see that D is an \bar{L} -connection. Extend D_\pm to $\Omega^{p,q}(V; \bar{L})$ by (3.2), then $D = D_+ + D_-$ is extended to $\Omega^*(V; \bar{L})$. We compute

$$D \circ D = (D_+ + D_-) \circ (D_+ + D_-) = D_+ \circ D_+ + [D_+, D_-] + D_- \circ D_- = 0$$

which implies the that V is a \mathbb{J}_+ -holomorphic bundle. \square

Lemma 3.11. *Let (V, D) be a \mathbb{J}_+ -holomorphic vector bundle. Then D_\mp define structures of \mathcal{A}_\pm -modules on the induced I_\pm -holomorphic vector bundles (V, D_\pm) .*

Proof: We prove the case of V_+ . The I_+ -holomorphic structure on V induces one on $V \otimes \mathcal{A}_+^* \cong V \otimes \bar{\ell}_-^*$, both of which coincide with D_+ (cf. Lemma 3.1 and (3.2)):

$$D_+ : C^\infty(V) \rightarrow C^\infty(V \otimes \bar{\ell}_+^*) \implies D_+ : C^\infty(V \otimes \bar{\ell}_-^*) \rightarrow C^\infty(V \otimes \bar{\ell}_-^* \otimes \bar{\ell}_+^*)$$

whose (local) kernels consist of (local) I_+ -holomorphic sections. For $D_- : C^\infty(V) \rightarrow C^\infty(V \otimes \bar{\ell}_-^*)$, the commutation relation $[D_+, D_-] = 0$ implies that when v is a local I_+ -holomorphic section of V , $D_-(v)$ is a local I_+ -holomorphic section of $V \otimes \bar{\ell}_-^*$. Thus, we obtain $\partial_{\mathcal{A}_+} : V \rightarrow V \otimes \mathcal{A}_+^*$. \square

Similar to Proposition 3.6, we have

Proposition 3.12. $\mathbb{H}^*(V; \mathcal{A}_+) \cong \mathbb{H}^*(V; \mathcal{A}_-) \cong H^*(V; \mathbb{J}_+)$. Moreover, the first page of the first spectral sequence $({}_I E_r^{p,q}, d_r)$ is

$${}_I E_1^{p,q} = H_{\bar{\partial}_-}^{0,q}(V \otimes \wedge^p \mathcal{A}_-^*) \text{ with } d_1 \text{ induced by } D_+$$

Similarly, the first page of the the second spectral sequence is

$${}_{II} E_1^{p,q} = H_{\bar{\partial}_+}^{0,q}(V \otimes \wedge^p \mathcal{A}_+^*) \text{ with } d'_1 \text{ induced by } D_-$$

Proof: The vertical and horizontal sequences in the double complex $(\Omega^{p,q}(V; \bar{L}), D_+, D_-)$ are respectively the resolutions of the holomorphic de Rham complexes of \mathcal{A}_\mp (cf. [32]). \square

Remark 3.13. As in Remark 3.7, Proposition 3.12 corresponds to Theorem 4.19 of [23].

Suppose now that V is an \mathcal{A}_+ -module (the case of \mathcal{A}_- -module is similar). Namely

- (1) V is I_+ -holomorphic, with the I_+ -holomorphic structure $\bar{\partial}_+ : C^\infty(V) \rightarrow C^\infty(V \otimes T_+^{0,1}M)$
- (2) together with a flat \mathcal{A}_+ -connection, i.e. $\partial_{\mathcal{A}_+} : V \rightarrow V \otimes \mathcal{A}_+^*$ and $\partial_{\mathcal{A}_+} \circ \partial_{\mathcal{A}_+} = 0$.

We recover the structure of a \mathbb{J}_+ -holomorphic vector bundle on V . Since $\bar{\ell}_-$ is the underlying smooth Lie algebroid of \mathcal{A}_+ and $\bar{\ell}_- \cong T_{0,1}^-M$ as smooth Lie algebroids, $\partial_{\mathcal{A}_+}$ induces an I_- -holomorphic structure $\bar{\partial}_-$ on V :

- (1) For a local holomorphic section v , $\bar{\partial}_- v := \partial_{\mathcal{A}_+} v$ under the isomorphism $\mathcal{A}_+ \cong T_{0,1}^-M$
- (2) For a local smooth function f , $\bar{\partial}_-(f v) := \bar{\partial}_- f \otimes v + f \bar{\partial}_- v$

This defines $\bar{\partial}_-$ for all local smooth sections of V . For the same f and v as above, we have

$$\bar{\partial}_- \circ \bar{\partial}_-(fv) = f\partial_{\mathcal{A}_+} \circ \partial_{\mathcal{A}_+} v = 0$$

i.e., V is an I_{\pm} -holomorphic bundle. By Lemma 3.10, we need to verify the commutation relation.

Lemma 3.14. *Extend $\bar{\partial}_{\pm}$ to D_{\pm} on $\Omega^{p,q}(V; \bar{L})$ using (3.2), then the commutation relation holds.*

Proof: Written in terms of covariant derivatives, the commutation relation becomes

$$D_{+,s}D_{-,t}(v) - D_{-,\bar{\delta}_{+,s}t}(v) - D_{-,t}D_{+,s}(v) + D_{+,\bar{\delta}_{-,t}s}(v) = 0 \text{ for } s \in \bar{\ell}_+, t \in \bar{\ell}_-$$

The equation $[\bar{\delta}_+, \bar{\delta}_-] = 0$ implies that the left hand side is tensorial in all variables. The situation can be simplified as following.

- (1) We only need to consider when v is a I_+ -holomorphic section of V , i.e. $D_+(v) = 0$.
- (2) We may take s to be I_- -holomorphic and t to be I_+ -holomorphic, i.e.

$$\bar{\delta}_{+,s}t = 0 \text{ and } \bar{\delta}_{-,t}s = 0$$

The remaining term is $D_{+,s}D_{-,t}(v) = D_{+,s}\partial_{\mathcal{A}_+,t}(v) = 0$. \square

The two Lemmata 3.11 and 3.14 imply that a complex vector bundle V is \mathbb{J}_+ -holomorphic iff it is an \mathcal{A}_+ -module iff it is an \mathcal{A}_- -module. Going through the construction, it is fairly clear that they are functorial. In summary, we obtain a slight refinement of the statement in [15] concerning equivalence of categories (Corollary 3.15 *loc. cit.*).

Proposition 3.15. *On a generalized Kähler manifold, the categories of \mathbb{J}_+ -holomorphic vector bundles and (locally free) \mathcal{A}_{\pm} -modules are isomorphic to each other.* \square

Remark 3.16. The correspondence of \mathbb{J}_+ -holomorphic vector bundles with \mathcal{A}_{\pm} -modules follows also from Lemma 4.16 of [23]. Our proof is basically an adaptation of theirs.

3.3. The case of \mathbb{J}_- . We discuss briefly the case of \mathbb{J}_- . As mentioned previously, the situation corresponds to basically reversing the complex structure I_- . The Lie algebroid decomposition is

$$L_- = \ell_+ \oplus \bar{\ell}_- \implies \wedge^* \bar{L}_- \cong \bigoplus_{p,q} \wedge^p \bar{\ell}_+^* \otimes \wedge^q \ell_-^*$$

which gives the isomorphism as linear spaces

$$\Omega^*(\bar{L}_-) \cong \bigoplus_{p,q} \Omega^{p,q}(\bar{L}_-), \text{ where } \Omega^{p,q}(\bar{L}_-) = \Omega^p(\bar{\ell}_+) \otimes \Omega^q(\ell_-)$$

Note that $\Omega^q(\ell_-) \cong \Omega_-^{q,0}(M)$. Instead of (3.1), we have

$$(3.3) \quad \bar{\delta}_+ := \bar{\partial}_+ + \bar{\partial}^+ : \Omega^{p,q}(\bar{L}_-) \rightarrow \Omega^{p+1,q}(\bar{L}_-) \text{ and } \delta_- := \partial_- + \partial^- : \Omega^{p,q}(\bar{L}_-) \rightarrow \Omega^{p,q+1}(\bar{L}_-)$$

where

$$\delta_{\mp} : \Omega^*(M) \xrightarrow{\nabla^{\pm}} \Omega^*(M) \otimes \Omega^1(M) \rightarrow \Omega^*(M) \otimes \Omega_{\mp}^{1,0}(M)$$

It then follows that $(\Omega^{p,q}(\bar{L}_-), \bar{\delta}_+, \delta_-)$ is a double complex and the associated spectral sequences computes $H^*(\bar{L}_-)$.

We obtain an I_+ -holomorphic Lie subalgebroid $\mathcal{B}_+ \subset \mathcal{E}_+$ from \bar{L}_- , via the reduction with respect to $\bar{\ell}_+$. The underlying smooth Lie algebroid is given by ℓ_- . The double complex $(\Omega^{p,q}(\bar{L}_-), \bar{\delta}_+, \delta_-)$ is a resolution of the holomorphic de Rham complex of \mathcal{B}_+ and thus we get

$$\mathbb{H}^*(\mathcal{B}_+) \cong H^*(\bar{L}_-)$$

Similarly, a \mathbb{J}_- -holomorphic vector bundle V corresponds to a \mathcal{B}_+ -module, and we have

$$\mathbb{H}^*(V; \mathcal{B}_+) \cong H^*(V; \mathbb{J}_-)$$

In the counterpart for I_- , we may either work with $-I_-$ or $-\mathbb{J}_-$. To stay in \mathcal{E}_- , we work with $-\mathbb{J}_-$ and the corresponding Lie algebroid L_- . Then, L_- induces an I_- -holomorphic Lie subalgebroid

$\mathcal{B}_- \subset \mathcal{E}_-$ via reduction with respect to $\bar{\ell}_-$. We have the similar double complex $(\Omega^{p,q}(L_-), \delta_+^{\prime\prime}, \bar{\delta}_-^{\prime\prime})$ and the isomorphism of cohomology groups

$$\mathbb{H}^*(\mathcal{B}_-) \cong H^*(L_-) \cong \overline{H^*(\bar{L}_-)}$$

For a \mathbb{J}_- -holomorphic vector bundle V , \bar{V} is now naturally a $-\mathbb{J}_-$ -holomorphic vector bundle and corresponds to a \mathcal{B}_- -module. We have

$$\mathbb{H}^*(\bar{V}; \mathcal{B}_-) \cong H^*(\bar{V}; -\mathbb{J}_-) \cong \overline{H^*(V; \mathbb{J}_-)}$$

Similar to Proposition 3.15, the category of \mathbb{J}_- -holomorphic vector bundles is isomorphic to that of the \mathcal{B}_+ -modules and \mathbb{C} -antilinearly isomorphic to that of the \mathcal{B}_- -modules.

3.4. Hodge decomposition. Recall that in the generalized Hodge decomposition associated to a generalized Kähler structure, $U_{-n,0} = C^\infty(U)$, where U is the canonical line bundle of \mathbb{J}_+ (cf. §2.5). We describe how the double complex $(\Omega^{p,q}(U; \bar{L}_+), D_+, D_-)$ for U is related to the generalized Hodge decomposition.

Again, let $L := L_+$ and $\bar{L} := \bar{L}_+$. The pairing $2\langle \cdot, \cdot \rangle$ defines the isomorphism $L \cong \bar{L}^*$:

$$\mathfrak{X} \mapsto 2\langle \mathfrak{X}, \bullet \rangle \text{ for } \mathfrak{X} \in L$$

which extends to isomorphisms $\wedge^* L \cong \wedge^* \bar{L}^*$. Let \mathcal{D} denote the differential on $C^\infty(\wedge^* L)$ induced by this identification, then for $\mathfrak{X} \in C^\infty(L)$ and $s, t \in C^\infty(\bar{L})$:

$$(3.4) \quad (\mathcal{D}\mathfrak{X})(s, t) = 2(a(s)\langle \mathfrak{X}, t \rangle - a(t)\langle \mathfrak{X}, s \rangle - \langle \mathfrak{X}, s * t \rangle) = -2\langle \mathfrak{X} * s, t \rangle$$

Lemma 3.17. *For $\mathfrak{X} \in C^\infty(L)$ and $\chi \in C^\infty(U)$, we have $\bar{\partial}_+^\gamma(\mathfrak{X} \cdot \chi) + \mathfrak{X} \cdot \bar{\partial}_+^\gamma \chi = \mathcal{D}\mathfrak{X} \cdot \chi$.*

Proof: By the decomposition of d_γ in (2.7), for $s, t \in C^\infty(\bar{L})$, we compute

$$\begin{aligned} (t \wedge s) \cdot (\bar{\partial}_+^\gamma(\mathfrak{X} \cdot \chi) + \mathfrak{X} \cdot \bar{\partial}_+^\gamma \chi) &= (t \wedge s) \cdot (d_\gamma(\mathfrak{X} \cdot \chi) + \mathfrak{X} \cdot d_\gamma \chi) \\ &= (t \wedge s) \cdot \mathcal{L}_\mathfrak{X} \chi = -t \cdot ((\mathfrak{X} * s) \cdot \chi) = -2\langle \mathfrak{X} * s, t \rangle \chi = (\mathcal{D}\mathfrak{X})(s, t) \chi \end{aligned}$$

Here $\mathcal{L}_\mathfrak{X}$ denotes the extended Lie derivative in the sense of Hu-Urbe [20]. Because $s \cdot \chi = t \cdot \chi = 0$, we have $(\mathcal{D}\mathfrak{X})(s, t) \chi = (t \wedge s) \cdot (\mathcal{D}\mathfrak{X}) \cdot \chi$. \square

Proposition 3.18. *For $\alpha \in C^\infty(\wedge^k L)$ and $\chi \in C^\infty(U)$, we have*

$$\bar{\partial}_+^\gamma(\alpha \cdot \chi) = \mathcal{D}\alpha \cdot \chi + (-1)^k \alpha \cdot d_\gamma \chi$$

Proof: Lemma 3.17 gives the case of $k = 1$. Suppose that the statement holds for $\alpha \in C^\infty(\wedge^k L)$. Let $\mathfrak{X} \in C^\infty(L)$ and we have

$$d_\gamma(\mathfrak{X} \cdot \alpha \cdot \chi) = \mathcal{L}_\mathfrak{X}(\alpha \cdot \chi) - \mathfrak{X} \cdot d_\gamma(\alpha \cdot \chi)$$

Taking the component in $C^\infty(\wedge^{k+2} L) \cdot C^\infty(U)$, we obtain

$$\begin{aligned} \bar{\partial}_+^\gamma((\mathfrak{X} \wedge \alpha) \cdot \chi) &= \alpha \cdot \mathcal{L}_\mathfrak{X} \chi - \mathfrak{X} \cdot \bar{\partial}_+^\gamma(\alpha \cdot \chi) \\ &= \alpha \cdot \mathcal{D}\mathfrak{X} \cdot \chi - \mathfrak{X} \cdot (\mathcal{D}\alpha \cdot \chi + (-1)^k \alpha \cdot d_\gamma \chi) \\ &= \mathcal{D}(\mathfrak{X} \wedge \alpha) \cdot \chi + (-1)^{k+1} (\mathfrak{X} \wedge \alpha) \cdot d_\gamma \chi \end{aligned}$$

The statement then follows by mathematical induction. \square

Theorem 3.19. $(\Omega^{p,q}(U; \bar{L}), D_+, D_-) \cong (U_{r,s}, \bar{\delta}_+^\gamma, \bar{\delta}_-^\gamma)$, where $r = p + q - n$ and $s = p - q$.

Proof: The groups are identified by the pairing $2\langle \cdot, \cdot \rangle$ on $\mathbb{T}M$ and the Clifford action:

$$\Omega^{p,q}(U; \bar{L}) = C^\infty(\wedge^p \bar{\ell}_+^* \otimes \wedge^q \bar{\ell}_-^* \otimes U) \cong C^\infty(\wedge^p \ell_+ \otimes \wedge^q \ell_-) \otimes U_{-n,0} \xrightarrow{\cong} U_{r,s}$$

The identification of the differentials follows from Proposition 3.18. \square

The generalized Hodge decomposition for $H_\gamma^*(M)$ in [12] then implies the following.

Corollary 3.20. $H^*(U; \mathbb{J}_+) \cong H_\gamma^*(M)$. \square

4. COURANT TRIVIALIZATION

Generalized Kähler geometry on K can be described explicitly using the *Courant trivialization* of $\mathbb{T}G := TG \oplus T^*G$ given in [1] (Remark 3.4, *loc. cit.*), where $G = K^{\mathbb{C}}$ is the complexified Lie group. We recall the relevant constructions here.

4.1. Notations. Let \mathfrak{g} be the Lie algebra of G , which is identified with the *right invariant* vector fields, then $\mathfrak{g} = \mathfrak{k} \otimes \mathbb{C}$. For $a \in \mathfrak{g}$, let X_a^r denote the right invariant vector field with value a at the identity $e \in G$, and respectively θ^r be the \mathfrak{g} -valued right invariant Cartan 1-form, i.e.

$$\theta^r(X_a^r) = a$$

The left invariant vector fields are denoted X_a^l for $a \in \mathfrak{g}$ and the left Cartan 1-form is denoted θ^l . The adjoint action relates the left and right invariant objects, for example, at $g \in G$

$$X_a^l(g) = X_{\text{Ad}_g a}^r(g) \text{ with } \text{Ad}_g a = L_{g*} R_{g*}^{-1} a$$

Let $F : \mathfrak{g}^{\otimes k} \rightarrow R$ be a \mathbb{C} -linear map to a \mathbb{C} -linear space R , then $F(\theta^r)$ is the R -valued right invariant k -form on G , defined by skew-symmetrization of F :

$$(F(\theta^r))(X_{a_1}^r, \dots, X_{a_p}^r) = \frac{1}{p!} \sum_{\sigma \in S_p} (-1)^\sigma F(a_{\sigma(1)}, \dots, a_{\sigma(p)})$$

Similarly, the left-invariant form $F(\theta^l)$ is defined. When G is semi-simple, the Killing form κ on \mathfrak{g} defines a bi-invariant Riemannian metric on K :

$$\sigma(X_a^r, X_b^r) = \kappa(a, b) = -\text{tr}(\text{ad}_a \circ \text{ad}_b)$$

and the bi-invariant Cartan 3-form γ is given by $\gamma = \kappa(\theta^r, [\theta^r, \theta^r])$, or more explicitly

$$\gamma(X_a^r, X_b^r, X_c^r) = \kappa(a, [b, c]) =: \Lambda(a, b, c)$$

In general, $G = G' \times T'$, where G' is semi-simple and T' is an abelian Lie group. A bi-invariant metric can be defined on K using the Killing form on \mathfrak{g}' and a non-degenerate bi-invariant form on \mathfrak{t}' . This is the case for Hopf surfaces mentioned previously.

Let $\mathbb{T}G = TG \oplus T^*G$, we have the Dorfman bracket defined by γ

$$(X + \xi) * (Y + \eta) = [X, Y] + \mathcal{L}_X \eta - \iota_Y d\xi + \iota_X \iota_Y \gamma$$

which defines the structure of Courant algebroid on $\mathbb{T}G$. The restriction to $\mathbb{T}K$ defines the generalized tangent bundle of K .

4.2. Linear algebra. Consider the isomorphism of linear spaces with quadratic forms:

$$(4.1) \quad \kappa_* : \mathfrak{d} = (\mathfrak{g} \oplus \mathfrak{g}, -\kappa \oplus \kappa) \rightarrow (\mathfrak{g} \oplus \mathfrak{g}^*, \langle, \rangle) : (a, a') \mapsto (a' - a, \kappa(a) + \kappa(a'))$$

which induces the isomorphism of Clifford algebras $\kappa_* : \text{Cl}(\mathfrak{d}) \xrightarrow{\cong} \text{Cl}(\mathfrak{g} \oplus \mathfrak{g}^*)$. Write $\hat{a} := (a, a') \in \overline{\mathfrak{d}}$, the natural spinor modules for these two Clifford algebras are

- (1) $\wedge^* \mathfrak{g}$ as $\text{Cl}(\mathfrak{g} \oplus \mathfrak{g}^*)$ -module: $(a, \xi) \circ v = a \wedge v + \iota_\xi v$ for $v \in \wedge^* \mathfrak{g}$
- (2) $\text{Cl}(\mathfrak{g}, \kappa)$ as $\text{Cl}(\mathfrak{d})$ -module: $\hat{a} \circ u = a' \cdot u - (-1)^{|u|} u \cdot a$ for $u \in \text{Cl}(\mathfrak{g}, \kappa)$

The *quantization map* $q : \wedge^* \mathfrak{g} \rightarrow \text{Cl}(\mathfrak{g}, \kappa)$ is the unique spinor module (iso)morphism with respect to κ_* such that $q(1) = 1$. By definition,

$$(4.2) \quad q(\kappa_* \hat{a} \circ v) = \hat{a} \circ q(v) \text{ for } v \in \wedge^* \mathfrak{g}$$

which implies that $q|_{\mathfrak{g}} = id_{\mathfrak{g}}$. Let $v \in \wedge^* \mathfrak{g}$, then $q(v) \in \text{Cl}(\mathfrak{g}, \kappa)$. The left and right Clifford multiplications in $\text{Cl}(\mathfrak{g}, \kappa)$ interact with q as following

$$a \cdot q(v) = q(a \wedge v + \iota_{\kappa(a)} v) \text{ and } (-1)^{|v|} q(v) \cdot a = q(a \wedge v - \iota_{\kappa(a)} v)$$

Switching \mathfrak{g} and \mathfrak{g}^* , we have the $\text{Cl}(\mathfrak{g} \oplus \mathfrak{g}^*)$ -module $\wedge^* \mathfrak{g}^*$:

$$(a, \xi) \circ \alpha = \iota_a \alpha + \xi \wedge \alpha \text{ for } \alpha \in \wedge^* \mathfrak{g}^*$$

Fix a nonzero element $\mu \in \wedge^{2n}\mathfrak{g}$. The *star map* $\star : \wedge^*\mathfrak{g}^* \rightarrow \wedge^{2n-*}\mathfrak{g}$ is the unique spinor module (iso)morphism with $\star 1 = \mu$. It follows that for $a \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$, we have $\star \xi = \iota_\xi \mu$ and

$$\star(\iota_a \alpha + \xi \wedge \alpha) = a \wedge \star \alpha + \iota_\xi(\star \alpha) \text{ for } \alpha \in \wedge^*\mathfrak{g}^*$$

We thus have the isomorphism of spinor modules:

$$q \circ \star : \wedge^*\mathfrak{g}^* \xrightarrow{\cong} \text{Cl}(\mathfrak{g}, \kappa)$$

Since the adjoint action of G preserves κ , it extends to the adjoint action on $\text{Cl}(\mathfrak{g}, \kappa)$. We lift the adjoint action $\text{Ad} : G \rightarrow \text{SO}(\mathfrak{g}, \kappa)$ to a group homomorphism $\tau : G \rightarrow \text{Spin}(\mathfrak{g}, \kappa) \subset \text{Cl}(\mathfrak{g}, \kappa)$, and write $\tau_g = \tau(g)$, then

$$\text{Ad}_g(u) = \tau_g \cdot u \cdot \tau_g^{-1} \text{ for } u \in \text{Cl}(\mathfrak{g}, \kappa)$$

This is possible because of G is connected and $\pi_1(G)$ is torsionless.

4.3. The trivializations. The left and right actions of G on itself define a left action of the *double group* $D = G \times G$ on G :

$$\hat{g} \circ h = g' h g^{-1} \text{ for } \hat{g} = (g, g') \in D$$

The adjoint action of G is induced by the diagonal embedding $G \hookrightarrow D$. Let $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ be the Lie algebra, then the infinitesimal action of $\hat{a} = (a, a') \in \mathfrak{d}$ is given by

$$X_{\hat{a}} = X_{a'}^r - X_a^l$$

It lifts to an extended action of \mathfrak{d} on $\mathbb{T}G$

$$\mathfrak{d} \rightarrow C^\infty(\mathbb{T}G) : \hat{a} \mapsto \mathfrak{X}_{\hat{a}} = X_{a'}^r - X_a^l + \kappa(\theta^r, a') + \kappa(\theta^l, a)$$

in the sense that the extended symmetry (cf. [20]) generated by $\mathfrak{X}_{\hat{a}}$ is precisely $X_{\hat{a}}$. Let $\hat{a} = (a, a')$ and $\hat{b} = (b, b') \in \mathfrak{d}$, direct computation shows

$$\langle \mathfrak{X}_{\hat{a}}, \mathfrak{X}_{\hat{b}} \rangle = \kappa(a', b') - \kappa(a, b) \text{ and } \mathfrak{X}_{\hat{a}} * \mathfrak{X}_{\hat{b}} = \mathfrak{X}_{[\hat{a}, \hat{b}]}$$

Thus we obtain the *Courant trivialization* (cf. [1])

$$G \times \mathfrak{d} \xrightarrow{\cong} \mathbb{T}G : \hat{a} \mapsto \mathfrak{X}_{\hat{a}}$$

which coincides with κ_* in (4.1) when restricted to the identity $e \in G$. The following will be useful.

Lemma 4.1 ([1]). *Suppose $\mathfrak{l} \subset \mathfrak{d}$ is an isotropic Lie subalgebra, then the subbundle L generated by $\mathfrak{X}_{\hat{a}}$ with $\hat{a} \in \mathfrak{l}$ is a Dirac structure, in particular is a Lie algebroid under the restriction of $*$. \square*

The isomorphism of Clifford algebras induced by κ_* extends to an isomorphism of Clifford bundles

$$(4.3) \quad G \times \text{Cl}(\mathfrak{d}) \rightarrow \text{Cl}(TG \oplus TG, -\sigma \oplus \sigma) \xrightarrow{\kappa_*} \text{Cl}(\mathbb{T}G) : (g, \hat{a}) \mapsto (X_a^l, X_{a'}^r)(g) \mapsto \mathfrak{X}_{\hat{a}}(g)$$

Since $\mu \in \wedge^{2n}\mathfrak{g}$ defines naturally a bi-invariant top multivector $\mu_G \in \Omega^*(G)$, we extend \star pointwisely to $\star : \wedge^*T^*G \rightarrow \wedge^*TG$. Define $\text{Ad}_{g*} = L_{g*}R_{g*}^{-1}$ on $\wedge^*T^*(G)$, then

$$\star \circ \text{Ad}_g = \text{Ad}_{g*} \circ \star : \wedge^*T^*G \rightarrow \wedge^*TG$$

The quantization map q induces the isomorphism

$$(4.4) \quad G \times \text{Cl}(\mathfrak{g}, \kappa) \rightarrow \text{Cl}(TG, \sigma) \xrightarrow{q^{-1}} \wedge^*TG : (g, u) \mapsto X_{u, \tau_g^{-1}}^r(g) \mapsto q^{-1}(X_{u, \tau_g^{-1}}^r(g))$$

where for $a_i \in \mathfrak{g}$, $u = a_1 \cdots a_k \in \text{Cl}(\mathfrak{g}, \kappa)$ we write $X_u^r := X_{a_1}^r \cdots X_{a_k}^r$. The isomorphism of spinor modules (4.2) implies that the Clifford action of terms in (4.3) on the respective term in (4.4) can be identified, which gives the isomorphism of the spinor modules.

Lemma 4.2 ([1], Proposition 4.2 a)). *The map $\mathcal{Q} : G \times \text{Cl}(\mathfrak{g}, \kappa) \rightarrow \wedge^*T^*G :$*

$$(g, u) \mapsto (q \circ \star)^{-1} \left(X_{u, \tau_g^{-1}}^r(g) \right) \text{ for } u \in \text{Cl}(\mathfrak{g}, \kappa)$$

is an isomorphism of spinor modules with respect to the isomorphism in (4.3). \square

4.4. **Differentials.** The adjoint action of G on $\text{Cl}(\mathfrak{g}, \kappa)$ can be extended to a D -action:

$$\hat{g} \circ u = \tau_{g'} \cdot u \cdot \tau_g^{-1}$$

On the other hand, the adjoint action of D on \mathfrak{d} induces a D -action on $\text{Cl}(\mathfrak{d}, -\kappa \oplus \kappa)$:

$$\text{Ad}_{\hat{g}}(a, a') = (\text{Ad}_g(a), \text{Ad}_{g'}(a')) = \left(\tau_g \cdot a \cdot \tau_g^{-1}, \tau_{g'} \cdot a' \cdot \tau_{g'}^{-1} \right)$$

Then these two actions are compatible:

$$(4.5) \quad \hat{g} \circ (\hat{a} \circ u) = \text{Ad}_{\hat{g}}(\hat{a}) \circ (\hat{g} \circ u)$$

The D -action on G induces the action on $\Omega^*(G)$ by push-forward:

$$\hat{g} \circ \alpha = L_{g'*} R_{g*}^{-1} \alpha$$

For $u \in \text{Cl}(\mathfrak{g}, \kappa)$, let $\alpha_u(g) := \mathcal{Q}(g, u) \in \Omega^*(G)$. Let $g' \in G$, then we compute

$$R_{h*}(\alpha_u(g)) = R_{h*}(q \circ \star)^{-1}(X_{u \cdot \tau_g^{-1}}^r(g)) = (q \circ \star)^{-1}(X_{u \cdot \tau_g^{-1}}^r(gh)) = \alpha_{u \cdot \tau_h}(gh)$$

$$L_{h*}(\alpha_u(g)) = \text{Ad}_{h*}(q \circ \star)^{-1}(X_{u \cdot \tau_g^{-1}}^r(gh)) = (q \circ \star)^{-1}(X_{\tau_h \cdot u \cdot \tau_g^{-1} \cdot \tau_h^{-1}}^r(hg)) = \alpha_{\tau_h \cdot u}(hg)$$

Thus, \mathcal{Q} is D -equivariant ([1] Proposition 4.2 c)), i.e.

$$\hat{g} \circ \alpha_u = \alpha_{\hat{g} \circ u} \text{ for } u \in \text{Cl}(\mathfrak{g}, \kappa)$$

The homomorphism τ induces an homomorphism $\tau' : \mathfrak{g} \rightarrow \text{Cl}(\mathfrak{g}, \kappa)$:

$$\tau'_a := \left. \frac{d}{dt} \right|_{t=0} \tau_{\exp(ta)} \text{ for } a \in \mathfrak{g}$$

which implies that

$$[a, b] = \text{ad}_a b = \tau'_a \cdot b - b \cdot \tau'_a \text{ and } \tau'_{[a, b]} = [\tau'_a, \tau'_b]$$

Let $\{e_i\}_{i=1}^{2n}$ and $\{e^i\}_{i=1}^{2n}$ be dual basis of \mathfrak{g} , e.g. a κ -orthonormal \mathbb{R} -basis $\{e_i = e^i\}_{i=1}^{2n}$ of \mathfrak{k} , and let

$$\Theta := q(\Lambda) = \frac{1}{6} \sum_{i, j, k=1}^{2n} \Lambda(e_i, e_j, e_k) e^i \cdot e^j \cdot e^k$$

then direct computation shows ([1], Meinrenken [26])

$$(4.6) \quad \tau'_a = \frac{1}{4} \sum_{i=1}^{2n} [a, e_i] \cdot e^i = -\frac{1}{4} [\Theta, a] \in \text{Cl}(\mathfrak{g}, \kappa)$$

The infinitesimal action of $\hat{a} = (a, a') \in \mathfrak{d}$ is given by the Clifford action of $\tau'_a := (\tau'_a, \tau'_{a'}) \in \text{Cl}(\overline{\mathfrak{d}})$

$$\tau'_a \circ u = \tau'_{a'} \cdot u - u \cdot \tau'_a \text{ for } u \in \text{Cl}(\mathfrak{g}, \kappa)$$

since $\tau'_{a'}$ is even. The infinitesimal action of $\hat{a} \in \mathfrak{d}$ on $\Omega^*(G)$ is given by ²

$$\mathcal{L}_{X_{\hat{a}}} \alpha = \left. \frac{d}{dt} \right|_{t=0} \exp(t\hat{a}) \circ \alpha$$

It then follows from D -equivariance of \mathcal{Q} that

$$(4.7) \quad \alpha_{\tau'_a \circ u} = \mathcal{L}_{X_{\hat{a}}} \alpha_u$$

From [1], $d^{\text{Cl}} := \frac{1}{4} [\Theta, \bullet]$ is the *Clifford differential* on $\text{Cl}(\mathfrak{g}, \kappa)$ (cf. [26] Chap. 6), i.e. $d^{\text{Cl}} \circ d^{\text{Cl}} = 0$.

Lemma 4.3 ([1] Proposition 4.2 b)). $\alpha_{d^{\text{Cl}} u} = d_{\gamma} \alpha_u$.

²We use push-forward to define the Lie derivative here.

Proof: Using Clifford action of $C^\infty(\mathbb{T}G)$ on $\Omega^*(G)$, we have (cf. [20])

$$\mathcal{L}_{X_{\hat{a}}} \alpha = -\mathfrak{X}_{\hat{a}} \circ d_\gamma \alpha - d_\gamma(\mathfrak{X}_{\hat{a}} \circ \alpha)$$

where $(X + \xi) \circ \alpha = \iota_X \alpha + \xi \wedge \alpha$ is the Clifford action of $C^\infty(\mathbb{T}G)$ on $\Omega^*(G)$. For the infinitesimal action on $\text{Cl}(\mathfrak{g}, \kappa)$, direct computation shows that

$$\tau'_{\hat{a}} \circ u = \tau'_{a'} \cdot u - u \cdot \tau'_{a'} = -\hat{a} \circ d^{\text{Cl}} u - d^{\text{Cl}}(\hat{a} \circ u)$$

The statement follows from (4.7) and that $d^{\text{Cl}}(d_\gamma)$ and \mathcal{Q} have opposite parities (cf. [1]). \square

5. LIE ALGEBRAIC GENERALIZED KÄHLER STRUCTURES

Notation: The notation $\mathfrak{l} \boxplus \mathfrak{l}' \subseteq \mathfrak{g} \oplus \mathfrak{g}'$ means that $\mathfrak{l} \subseteq \mathfrak{g}$ and $\mathfrak{l}' \subseteq \mathfrak{g}'$, while $\mathfrak{p} \oplus \mathfrak{p}' \subseteq \mathfrak{V}$ means that \mathfrak{p} and \mathfrak{p}' are both subspaces of \mathfrak{V} .

Let $2n = \dim_{\mathbb{R}} \mathfrak{k}$, then $2n = \dim_{\mathbb{C}} \mathfrak{g}$. Let \mathfrak{t} be a Cartan subalgebra of \mathfrak{k} then $\mathfrak{h} = \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}$ is a Cartan subalgebra of \mathfrak{g} . Let R be the set of roots with respect to \mathfrak{h} . Since \mathfrak{k} is a compact Lie algebra, any root $\alpha \in R$ is purely imaginary on \mathfrak{t} . In particular, when \mathfrak{k} is semisimple, the Satake diagram of \mathfrak{k} is simply the Dynkin diagram of \mathfrak{g} with all vertices painted black.

Let \mathfrak{b} be a Borel subalgebra containing \mathfrak{h} and $R_+ \subset R$ (respectively R_-) be the corresponding set of positive (respectively negative) roots. We then have the triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_- \text{ where } \mathfrak{n}_{\pm} = \bigoplus_{\alpha \in R_{\pm}} \mathfrak{g}_{\alpha}$$

When \mathfrak{k} is semisimple, this is an orthogonal decomposition with respect to κ . In general, let \mathfrak{z} be the center of \mathfrak{g} then we can decompose \mathfrak{h} furthermore as

$$\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{z} \text{ where } \mathfrak{h}_0 = \mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]$$

It then follows that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{z}$, and we extend κ from the semisimple part $[\mathfrak{g}, \mathfrak{g}]$ to \mathfrak{g} such that this decomposition is orthogonal.

There are root vectors $a_{\alpha} \in \mathfrak{g}_{\alpha}$ for $\alpha \in R$ (cf. Humphreys [21]) such that

$$[a_{\alpha}, a_{\beta}] = a_{\alpha+\beta} \text{ for } \alpha \pm \beta \neq 0, \text{ and } \kappa(a_{\alpha}, a_{-\alpha}) = 1$$

Since $[h, a_{\alpha}] = \alpha(h)a_{\alpha}$ for any $h \in \mathfrak{t}$, we see that $\mathfrak{g}_{\alpha} \cap \mathfrak{k} = \{0\}$. Let $h_{\alpha} = [a_{\alpha}, a_{-\alpha}]$ for $\alpha \in R_+$, then $\kappa(h_{\alpha}, h) = \alpha(h)$ for all $h \in \mathfrak{h}$. It follows that

$$h_{\alpha} \in i\mathfrak{t} \subset \mathfrak{h} \text{ for all } \alpha \in R$$

5.1. Lagrangian subalgebras. The classification of complex Lagrangian subalgebra of in \mathfrak{d} (with respect to $-\kappa \oplus \kappa$) is obtained by Karolinsky [22]. We recall the description of a representative for each orbit of adjoint action of D on the space of Lagrangian subalgebras of \mathfrak{d} , as given in Evens-Lu [8]. Let $\mathfrak{b} \supset \mathfrak{h}$ be a Borel subalgebra of \mathfrak{g} and R_+ the corresponding set of positive roots. Let $\Gamma \subset R_+$ be the set of simple roots. For any $P \subseteq \Gamma$, we set

$$[P] := R \cap \text{Span}_{\mathbb{C}}\{\alpha \in P\}$$

Consider the corresponding subalgebra $\mathfrak{g}_P \subset \mathfrak{g}$ defined by the sub-diagram of the Dynkin diagram of \mathfrak{g} with vertices in P , then $\mathfrak{m}_P := \mathfrak{g}_P + \mathfrak{h}$ decomposes as $\mathfrak{m}_P = \mathfrak{g}_P \oplus \mathfrak{z}_P$, where \mathfrak{z}_P is the center of \mathfrak{m}_P . We have also the nilpotent subalgebras

$$\mathfrak{n}_P := \bigoplus_{\alpha \in R_+ \setminus [P]} \mathfrak{g}_{\alpha} \text{ and } \mathfrak{n}_P^- := \bigoplus_{\alpha \in R_+ \setminus [P]} \mathfrak{g}_{-\alpha}$$

A *generalized Belavin-Drinfeld triple* (P, P', π) consists of two subsets $P, P' \subseteq \Gamma$ and an *isometry* $\pi : P \rightarrow P'$, i.e. an isomorphism such that

$$\kappa(h_{\alpha}, h_{\beta}) = \kappa(h_{\pi\alpha}, h_{\pi\beta})$$

It induces a unique Lie algebra isomorphism $\psi_{\pi} : \mathfrak{g}_P \rightarrow \mathfrak{g}_{P'}$ such that

$$\psi_{\pi}(a_{\alpha}) = a_{\pi\alpha} \text{ for all } \alpha \in P$$

Lemma 5.1 ([8], Theorem 2.16). *Each orbit of the adjoint action of D on the space of Lagrangian subalgebras of \mathfrak{d} contains exactly one element of the form*

$$\mathfrak{l}(\pi, F) := F \oplus (\mathfrak{n}_P \oplus \mathfrak{n}_{P'}^-) \oplus L^\pi$$

where (P, P', π) is a generalized Belavin-Drinfeld triple, $F \subset \mathfrak{z}_P \oplus \mathfrak{z}_{P'}$ is a Lagrangian subspace, and L^π is the graph of ψ_π . \square

5.2. Invariant complex structures. Since $\mathfrak{g} = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$, a complex structure J on \mathfrak{k} corresponds to an n -dimensional complex subspace $\mathfrak{l}_J \subset \mathfrak{g}$ such that $\mathfrak{l}_J \cap \mathfrak{k} = \{0\}$. We say that J is *integrable* if \mathfrak{l}_J is a complex Lie subalgebra, called a *Samelson subalgebra* (cf. Samelson [29]). In this case, the corresponding invariant almost complex structures on K with value J at $e \in K$ are integrable.

Pittie [28] showed that there is a unique Borel subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$ such that $\mathfrak{l}_J \subseteq \mathfrak{b}$. Furthermore, an integrable complex structure J on \mathfrak{k} is in 1-1 correspondence to the pairs (\mathfrak{b}, J') where

- (1) $\mathfrak{b} \subset \mathfrak{g}$ is a Borel subalgebra, and $\mathfrak{h}' \in \mathfrak{b}$ is the corresponding complex Cartan subalgebra
- (2) J' is a complex structure on the corresponding real Cartan subalgebra $\mathfrak{t}' = \mathfrak{h}' \cap \mathfrak{k}$

We say that \mathfrak{b} or \mathfrak{t}' are the Borel subalgebra or real Cartan subalgebra *corresponding to the complex structure J* .

Proposition 5.2 ([28] Proposition 2.6, Corollary 2.7.1). *The space of (left) invariant σ -orthogonal integrable complex structures on K is isomorphic to $O(\mathfrak{h})/U(\mathfrak{h}) \times K/T$, and the moduli space up to $\text{Aut}(K)$ is isomorphic to $F \backslash O(\mathfrak{h})/U(\mathfrak{h})$, where F is the outer automorphism group of K . \square*

Let J be an integrable complex structure on \mathfrak{k} and let $\mathfrak{t}_{1,0} \subset \mathfrak{h}$ be the i -eigensubspace, then

$$\mathfrak{l}_J = \mathfrak{n}_+ \oplus \mathfrak{t}_{1,0} \text{ and } \bar{\mathfrak{l}}_J = \mathfrak{n}_- \oplus \mathfrak{t}_{0,1}$$

We note that the complex structure J on \mathfrak{k} can also be induced from the exact sequence

$$(5.1) \quad 0 \rightarrow \bar{\mathfrak{l}}_J \rightarrow \mathfrak{g} \rightarrow \mathfrak{k} \rightarrow 0$$

From now on, we assume that choices are made such that J is orthogonal with respect to κ on \mathfrak{k} . It implies that \mathfrak{l}_J as well as $\bar{\mathfrak{l}}_J$ are isotropic subspaces of \mathfrak{g} , with respect to κ .

Let J_\pm be two integrable complex structures on \mathfrak{k} . We denote by I_+ (respectively I_-) the right invariant (respectively left invariant) complex structure on K defined by J_+ (respectively J_-), e.g.

$$I_+(X_a^r) = X_{J_+a}^r \text{ and } I_-(X_a^l) = X_{J_-a}^l$$

It follows that the holomorphic tangent bundle of I_+ (respectively I_-) is spanned by X_a^r (respectively X_a^l) for $a \in \mathfrak{l}_+ := \mathfrak{l}_{J_+}$ (respectively $a \in \mathfrak{l}_- := \mathfrak{l}_{J_-}$). Unless K is abelian, i.e. a torus, the right invariant complex structure I_+ is not left invariant. On the other hand, both I_\pm are invariant under the action of the maximal torus generated by \mathfrak{t} , since

$$[\mathfrak{t}, \mathfrak{l}_J] \subseteq \mathfrak{l}_J$$

The following result will be useful later.

Lemma 5.3. *σ is Gauduchon with respect to both I_+ and I_- .*

Proof: We verify the $+$ -case. Let $\omega_+ = \sigma I_+$ be the Kähler form and $d\text{vol}_\sigma$ denote the invariant volume form with respect to σ . Since both I_+ and ω_+ are right-invariant, $dd_+^c(\omega_+^{n-1})$ is a right-invariant top form, i.e. $dd_+^c(\omega_+^{n-1}) = A d\text{vol}_\sigma$ for some $A \in \mathbb{R}$. Since K is compact

$$0 = \int_K dd_+^c(\omega_+^{n-1}) = \int_K A d\text{vol}_\sigma$$

Thus $A = 0$ and $dd_+^c(\omega_+^{n-1}) = 0$, i.e. σ is Gauduchon with respect to I_+ . \square

It follows that for any I_\pm -holomorphic vector bundle V on K , we have the notion of \pm -degrees :

$$\deg_\pm(V) := \frac{i}{2} \frac{\int_M F_\pm \wedge \omega_\pm^{n-1}}{\int_M \omega_\pm^{n-1}}$$

where F_\pm are the curvatures of the Chern connections with respect to any Hermitian metric on V . For $\alpha \in (0, 1)$, we have the α -degree ([19]):

$$\deg_\alpha(V) = \alpha \deg_+(V) + (1 - \alpha) \deg_- V$$

The \pm -slope μ_\pm and α -slope μ_α are defined as quotients of the corresponding degrees by the rank.

5.3. Bismut connections. The Bismut connections ∇^\pm on K for the Hermitian structures given by I_\pm has respectively torsions $\pm\gamma$. Recall that I_\pm is parallel in ∇^\pm respectively. They can also be defined from the Dorfman bracket as following. Let $X, Y \in TK$, then

$$\nabla_X^\pm Y = a \{ [X \mp \sigma(X) * (Y \pm \sigma(Y))]^\pm \} = \nabla_X Y \pm \frac{1}{2} \sigma^{-1} \iota_X \iota_Y \gamma$$

where a is the projection to TK and $[\bullet]^\pm$ denote the projections to C_\pm .

Lemma 5.4. *K is flat with respect to both ∇^\pm , i.e. TK can be trivialized by ∇^\pm -horizontal sections.*

Proof: We prove the $+$ -case. For $a, b' \in \mathfrak{k}$, let $d_1 = (a, 0)$ and $d_2 = (0, b') \in \mathfrak{d}$, then $[d_1, d_2] = 0$. It follows that

$$\mathfrak{X}_{d_1} * \mathfrak{X}_{d_2} = \mathfrak{X}_{[d_1, d_2]} = 0$$

Since $\mathfrak{X}_{d_1} = -X_a^l + \kappa(\theta^l, a) \in C_-$ and $\mathfrak{X}_{d_2} = X_{b'}^r + \kappa(\theta^r, b') \in C_+$, it implies that $\nabla_{X_a^l}^+ X_{b'}^r = 0$. The result follows from the fact that the left invariant vector fields span TK . \square

Corollary 5.5. *Suppose that $\alpha \in \Omega^p(K)$ is a right (respectively, left) invariant form, then $\nabla^+ \alpha = 0$ (respectively, $\nabla^- \alpha = 0$).*

Proof: We prove the case of right invariant forms. Let X_i be right invariant vector fields, for $i = 1, \dots, p$, then for any vector field X on K , we have

$$X(\alpha(X_1, \dots, X_p)) = (\nabla_X^+ \alpha)(X_1, \dots, X_p) + \sum_{i=1}^p (-1)^p \alpha(\nabla_X^+ X_i, X_1, \dots, \hat{X}_i, \dots, X_p)$$

The lefthand side vanishes because right invariant functions are constants. The terms in the summation vanish because all the covariant derivatives vanish by Lemma 5.4. Thus, $\nabla^+ \alpha = 0$. \square

5.4. Generalized Kähler structures. By Lemma 5.1, a complex Lagrangian subalgebra $\mathfrak{L} \subset \mathfrak{d}$ is of the form

$$(5.2) \quad \mathfrak{L} = \text{Ad}_{(g, g')} \mathfrak{l}(\pi, F) = \text{Ad}_{(g, g')} F \oplus (\text{Ad}_g \mathfrak{n}_P \oplus \text{Ad}_{g'} \mathfrak{n}_{P'}^-) \oplus \text{Ad}_{(g, g')} L^\pi$$

Lemma 4.1 implies that it defines a complex Dirac structure $L \subset \mathbb{T}_\mathbb{C} K$ via Courant trivialization. If furthermore $\mathfrak{L} \cap \mathfrak{k} \oplus \mathfrak{k} = \{0\}$, the corresponding L defines a generalized complex structure \mathbb{J} on K .

Definition 5.6. *A generalized complex structure on K defined via the Courant trivialization by a complex Lagrangian subalgebra $\mathfrak{L} \subset \mathfrak{d}$ satisfying $\mathfrak{L} \cap (\mathfrak{k} \oplus \mathfrak{k}) = \{0\}$ is called a Lie algebraic generalized complex structure. A generalized Kähler structure $(K, \gamma; \mathbb{J}_\pm, \sigma)$ is a Lie algebraic generalized Kähler structure if both \mathbb{J}_\pm are Lie algebraic generalized complex structures.*

Lemma 5.7. *Let $(K, \gamma; \mathbb{J}_\pm, \sigma)$ be a Lie algebraic generalized Kähler structure and $\mathfrak{L}_\pm \subset \mathfrak{d}$ be the corresponding complex Lagrangian subalgebra defining \mathbb{J}_\pm . Then there exist two (not necessarily distinct) Samelson subalgebras $\mathfrak{l}_\pm \subset \mathfrak{g}$ such that $\mathfrak{L}_+ = \mathfrak{l}_- \boxplus \mathfrak{l}_+$ and $\mathfrak{L}_- = \bar{\mathfrak{l}}_- \boxplus \mathfrak{l}_+$.*

Proof: Let L_{\pm} be the i -eigensubbundles of \mathbb{J}_{\pm} respectively, then we have $L_+ = \ell_+ \oplus \ell_-$ and $L_- = \ell_+ \oplus \bar{\ell}_-$ (cf. §2.1), where $\ell_{\pm} \subset C_{\pm}$. Via the Courant trivialization, by (4.1), ℓ_+ (respectively ℓ_-) is defined by a subspace ℓ'_+ of $0 \oplus \mathfrak{g}$ (respectively a subspace ℓ'_- of $\mathfrak{g} \oplus 0$). Thus, we have, for example

$$\mathfrak{L}_+ = \ell'_- \oplus \ell'_+ \subset \mathfrak{g} \oplus \mathfrak{g}$$

In the general form (5.2), since the adjoint action of D does not affect splitness, this implies that

- (1) L^{π} is trivial, i.e. the generalized Belavin-Drinfeld triple is trivial
- (2) \mathfrak{n}_P (respectively $\mathfrak{n}_{P'}$) is the sum of positive (respectively negative) root spaces
- (3) $F = F_- \oplus F_+$ where $F_{\pm} \subset \mathfrak{h}$ are complex Lagrangian subspaces of \mathfrak{h}

This proves the lemma. \square

Thus, the space of Lie algebraic generalized Kähler structures can be identified with the space of pairs of Samelson subalgebras, which admits the natural adjoint action by $K \times K$. Let \mathfrak{b}_{\pm} denote the unique Borel subalgebras that contain \mathfrak{l}_{\pm} respectively, then for any pair $(\mathfrak{l}_+, \mathfrak{l}_-)$ of Samelson subalgebras, there exists $(g, g') \in D$ such that $\text{Ad}_{g'} \mathfrak{b}_+ = \text{Ad}_g \mathfrak{b}_-$.

Definition 5.8. An induced generalized Kähler structure on K is a Lie algebraic generalized Kähler structure with $\mathfrak{b}_+ = \mathfrak{b}_-$. It is a canonical generalized Kähler structure if $\mathfrak{l}_+ = \mathfrak{l}_-$. Two Lie algebraic generalized Kähler structures are algebraically equivalent if the corresponding pairs of Samelson subalgebras are related by the adjoint action of D . They are geometrically equivalent if the corresponding pairs of Samelson subalgebras are related by the adjoint action of $K \times K$.

Lemma 5.9. Geometrically equivalent Lie algebraic generalized Kähler structures are isomorphic. Any Lie algebraic generalized Kähler structure on K is algebraically equivalent to an induced generalized Kähler structure.

Proof: Let $(g', g) \in K \times K$, define $(\mathfrak{l}_+, \mathfrak{l}_-) = (\text{Ad}_{g'} \mathfrak{l}'_+, \mathfrak{l}_-)$. For the corresponding invariant complex structures, we have

$$I_+ = \text{Ad}_{g'^*} I'_+ = L_{g'^*} I'_+$$

Since I_- is left invariant, we see that $(I_+, I_-) = L_{g'^*} (I'_+, I_-)$, i.e. $L_{g'}$ induces an isomorphism of the bi-Hermitian structures. Similarly, we see that when $(\mathfrak{l}_+, \mathfrak{l}_-) = (\mathfrak{l}'_+, \text{Ad}_g \mathfrak{l}_-)$, the isomorphism is induced by R_g . The second statement is straightforward. \square

The Lie algebraic generalized complex structures can be explicitly written down. In terms of I_{\pm} , for $a, a' \in \mathfrak{k}$, we have:

$$\begin{aligned} \mathbb{J}_+ (X_{a'}^r - X_a^l + \sigma(X_{a'}^r + X_a^l)) &= I_+ X_{a'}^r - I_- X_a^l + \sigma(I_+ X_{a'}^r + I_- X_a^l) \\ \mathbb{J}_- (X_{a'}^r - X_a^l + \sigma(X_{a'}^r + X_a^l)) &= I_+ X_{a'}^r + I_- X_a^l + \sigma(I_+ X_{a'}^r - I_- X_a^l) \end{aligned}$$

Let

$$\ell_+ = \text{Span}_{\mathbb{C}} \{\mathfrak{X}_{0,a} : a \in \mathfrak{l}_+\} \text{ and } \ell_- = \text{Span}_{\mathbb{C}} \{\mathfrak{X}_{a,0} : a \in \mathfrak{l}_-\}$$

then the i -eigensubbundles are

$$L_+ = \ell_+ \oplus \ell_- = \text{Span}_{\mathbb{C}} \{\mathfrak{X}_{a,a'} : (a, a') \in \mathfrak{l}_- \oplus \mathfrak{l}_+\} \text{ and } L_- = \ell_+ \oplus \bar{\ell}_- = \text{Span}_{\mathbb{C}} \{\mathfrak{X}_{a,a'} : (a, a') \in \bar{\mathfrak{l}}_- \oplus \mathfrak{l}_+\}$$

Lemma 5.10. Let $(K, \gamma; \mathbb{J}_{\pm}, \sigma)$ be a canonical generalized Kähler structure. Let $T \subset K$ be the maximal torus generated by the real Cartan subalgebra \mathfrak{t} (corresponding to $J = J_+ = J_-$). Then, the restriction of $(K, \gamma; \mathbb{J}_{\pm}, \sigma)$ on T is an invariant Kähler structure.

Proof: For $a \in \mathfrak{t}$ and $h \in T$, we have $X_a^r(h) = X_a^l(h) =: X_a(h)$. Since J preserves \mathfrak{t} , we have $I_{\pm} X_a = X_{Ja}$. It follows that when restricted to T , we have

$$\mathbb{J}_+ (X_{a'-a} + \sigma(X_{a'+a})) = X_{J(a'-a)} + \sigma(X_{J(a'+a)}) \text{ and } \mathbb{J}_- (X_{a'-a} + \sigma(X_{a'+a})) = X_{J(a'+a)} + \sigma(X_{J(a'-a)})$$

It's now clear that \mathbb{J}_+ is defined by the invariant complex structure $I := I_{\pm}|_T$ and \mathbb{J}_- is defined by the invariant symplectic structure $\omega := \omega_{\pm}|_T$. They define an invariant Kähler structure. \square

As invariant complex structures are determined by the restriction to a maximal torus up to group isomorphisms (cf. [28]), the same holds for canonical generalized Kähler structures.

Corollary 5.11. *Up to group automorphisms, the canonical generalized Kähler structures on K are determined by the invariant Kähler structures on a maximal torus T .* \square

We may apply the construction directly to the even dimensional torus T and obtain generalized Kähler structures on it. Since T is abelian, the resulting generalized Kähler structures are invariant under T action, which will be called the *invariant generalized Kähler structures* on T . Corollary 5.11 generalizes to the statement about induced generalized Kähler structures.

Proposition 5.12. *Up to group automorphisms, induced generalized Kähler structures on K are determined by invariant generalized Kähler structures on a maximal torus T .* \square

Since the left and right invariant complex structures are invariant under the actions of their respective maximal tori generated by $\mathfrak{t}_\pm \subseteq \mathfrak{b}_\pm$, the resulting generalized Kähler structure is invariant under the action of a subtorus, i.e. the intersection. In this sense, induced structures carry the maximal amount of symmetries.

Proposition 5.13. *An induced generalized Kähler structure is invariant under the action of a maximal torus, which induces naturally on the quotient the homogeneous Kähler structure.* \square

5.5. Canonical bundles. We have first the vanishing result for semi-simple Lie groups.

Proposition 5.14. *$H^*(U_\pm; \mathbb{J}_\pm) = 0$ when K is a semi-simple Lie group.*

Proof: It's well-known that the twisted de Rham cohomology $H_\gamma^*(K)$ is identically trivial for semi-simple K , e.g. [9]. The result follows from Corollary 3.20. \square

In general, the \mathbb{J}_\pm -canonical bundles on K , using \mathcal{Q} in Lemma 4.2 (cf. [1]), are determined by the corresponding spinor lines in $\text{Cl}(\mathfrak{g}, \kappa)$. Let $\bar{\mathfrak{p}} \subseteq \mathfrak{t}_{-,0,1}$ be a subspace such that

$$\bar{\mathfrak{m}} := \bar{\mathfrak{l}}_+ + \bar{\mathfrak{l}}_- = \bar{\mathfrak{l}}_+ \oplus \bar{\mathfrak{p}} \subseteq \mathfrak{g}$$

Let $\{\bar{b}_j\}_{j=1}^{n+s}$ be a basis of $\bar{\mathfrak{m}}$ such that

$$(5.3) \quad \{\bar{b}_j\}_{j=1}^n \text{ is a basis of } \bar{\mathfrak{l}}_+, \{\bar{b}_j\}_{j=s+1}^{n+s} \text{ is a basis of } \bar{\mathfrak{l}}_-, \text{ and } \{\bar{b}_j\}_{j=n+1}^{n+s} \text{ is a basis of } \bar{\mathfrak{p}}$$

and define

$$(5.4) \quad u_+ = \bar{b}_1 \cdot \dots \cdot \bar{b}_{n+s} \in \text{Cl}(\mathfrak{g}, \kappa)$$

Similarly we can define u_- from $\bar{\mathfrak{l}}_+ + \bar{\mathfrak{l}}_- \subseteq \mathfrak{g}$.

Lemma 5.15. *u_+ as given above is a pure spinor defining $\bar{\mathfrak{l}}_- \boxplus \bar{\mathfrak{l}}_+ \subset (\mathfrak{d}, -\kappa \oplus \kappa)$, and the \mathbb{J}_+ -canonical bundle U_+ is generated by $\chi_+ := \mathcal{Q}(u_+) \in \Omega^*(K)$. The type of \mathbb{J}_+ (respectively of \mathbb{J}_-) has the same parity as $\dim_{\mathbb{C}} \bar{\mathfrak{l}}_+ \cap \bar{\mathfrak{l}}_-$ (respectively as $\dim_{\mathbb{C}} \bar{\mathfrak{l}}_+ \cap \bar{\mathfrak{l}}_-$).*

Proof: Left as an exercise for the reader. \square

For an induced generalized Kähler structures, where a Borel subalgebra \mathfrak{b} contains both \mathfrak{l}_\pm , the spinor u_+ admits a more explicit description. Let

$$\mathfrak{b} = \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha \oplus \mathfrak{h}$$

be the decomposition of the Borel subalgebra \mathfrak{b} into the positive root spaces \mathfrak{g}_α and the Cartan subalgebra \mathfrak{h} . Correspondingly,

$$\bar{\mathfrak{l}}_\pm = \bigoplus_{\alpha \in R_+} \mathfrak{g}_{-\alpha} \oplus \mathfrak{t}_{\pm,0,1} \text{ and } \bar{\mathfrak{p}} \subseteq \mathfrak{h}$$

Let $\{\bar{h}_i : \bar{h}_i \in \mathfrak{t}_{+,0,1}\}$ be a basis of $\mathfrak{t}_{+,0,1}$ and $\{\bar{p}_1, \dots, \bar{p}_s\}$ be a basis for $\bar{\mathfrak{p}}$. Choose $0 \neq a_{-\alpha} \in \mathfrak{g}_{-\alpha}$, then

$$\{\bar{b}_j\} := \{\bar{h}_i, a_{-\alpha}, \bar{p}_k\} \text{ is a basis of } \bar{\mathfrak{m}}$$

and satisfies the conditions in (5.3) and defines correspondingly the spinor u_+ in this case.

By Lemma 5.9 we have

Corollary 5.16. *The \mathbb{J}_+ -canonical bundle of a Lie algebraic generalized Kähler structure is of the form $\langle \mathcal{Q}(\hat{g} \circ u_+) \rangle$, where $\hat{g} \in D$ and u_+ is a spinor for an induced structure, as described above. \square*

The \mathbb{J}_+ -holomorphic structure on U_+ is given by the twisted differential d_γ on $C^\infty(U_+) \subset \Omega^*(M)$ (cf. §2.5, [11]), which corresponds to the Clifford differential d^{Cl} under the Courant trivialization (cf. Lemma 4.3, [1]). Let $\{b_j\}_{j=1}^n$ be the basis for \mathfrak{l}_+ dual to $\{\bar{b}_j\}_{j=1}^n$, i.e.

$$\{b_j\}_{j=1}^n = \{h_j : h_j \in \mathfrak{t}_{+,1,0}\} \cup \{a_\alpha \in \mathfrak{g}_\alpha : \alpha \in R_+\}$$

such that $\kappa(b_i, \bar{b}_j) = \delta_{ij}$. Let $h_\alpha = [a_\alpha, a_{-\alpha}] \in \mathfrak{h}$, then we have $\kappa(h_\alpha, h) = \alpha(h)$ for all $h \in \mathfrak{h}$.

$$u_{\mathfrak{l}_+} = \bar{b}_1 \cdot \dots \cdot \bar{b}_n \text{ and } u_{\mathfrak{p}} = \bar{p}_1 \cdot \dots \cdot \bar{p}_s$$

then $u_+ = u_{\mathfrak{l}_+} \cdot u_{\mathfrak{p}}$. We have the *Weyl vector*³

$$\rho = \frac{1}{2} \sum_{\alpha \in R_+} h_\alpha \in \mathfrak{h}$$

We note that the norm of the Weyl vector with respect to κ does not depend on the choice of the Borel subalgebra \mathfrak{b} , and will be denoted $|\rho|$.

Lemma 5.17. $d^{\text{Cl}}u_{\mathfrak{l}_+} = \frac{1}{2}(-\rho, \rho) \circ u_{\mathfrak{l}_+}$.

Proof: Let $\mathfrak{l} := \mathfrak{l}_+$. In the basis $\{b_j, \bar{b}_j\}_{j=1}^n$, write $b_{ij\bar{k}} := b_i \cdot b_j \cdot \bar{b}_k$, then Θ becomes

$$\Theta = \frac{1}{6} \sum_{i,j,k} \Lambda(b_i, b_j, \bar{b}_k)(b_{i\bar{j}k} + b_{k\bar{i}j} - b_{\bar{i}k\bar{j}}) + \frac{1}{6} \sum_{i,j,k} \Lambda(\bar{b}_i, \bar{b}_j, b_k)(b_{ij\bar{k}} + b_{\bar{k}ij} - b_{i\bar{k}j})$$

The terms in the first sum annilate $u_{\mathfrak{l}}$. The terms in the second sum give

$$\begin{aligned} [b_{ij\bar{k}}, u_{\mathfrak{l}}] &= -(-1)^n u_{\mathfrak{l}} \cdot b_i \cdot b_j \cdot \bar{b}_k = (-1)^n 2u_{\mathfrak{l}} \cdot (\delta_{ik}b_j - \delta_{jk}b_i) \\ [b_{\bar{k}ij}, u_{\mathfrak{l}}] &= \bar{b}_k \cdot b_i \cdot b_j \cdot u_{\mathfrak{l}} = 2(\delta_{ik}b_j - \delta_{jk}b_i) \cdot u_{\mathfrak{l}} \\ [b_{i\bar{k}j}, u_{\mathfrak{l}}] &= b_i \cdot \bar{b}_k \cdot b_j \cdot u_{\mathfrak{l}} - (-1)^n u_{\mathfrak{l}} \cdot b_i \cdot \bar{b}_k \cdot b_j = 2\delta_{jk}b_i \cdot u_{\mathfrak{l}} - (-1)^n 2u_{\mathfrak{l}} \cdot \delta_{ik}b_j \end{aligned}$$

It follows that

$$\begin{aligned} d^{\text{Cl}}u_{\mathfrak{l}} &= \frac{1}{4} [\Theta, u_{\mathfrak{l}}] = \frac{1}{4} \sum_{i,j} \Lambda(\bar{b}_i, \bar{b}_j, b_i)(b_j \cdot u_{\mathfrak{l}} + (-1)^n u_{\mathfrak{l}} \cdot b_j) \\ &= \frac{1}{4} \sum_{i,j} \kappa([b_i, \bar{b}_i], \bar{b}_j)(b_j \cdot u_{\mathfrak{l}} + (-1)^n u_{\mathfrak{l}} \cdot b_j) = \frac{1}{2}(\rho \cdot u_{\mathfrak{l}} + (-1)^n u_{\mathfrak{l}} \cdot \rho) \end{aligned}$$

The last expression coincides with the Clifford action of $\frac{1}{2}(-\rho, \rho) \in \text{Cl}(\bar{\mathfrak{d}})$ on $u_{\mathfrak{l}}$. \square

Corollary 5.18. *Write $\mathfrak{l} := \mathfrak{l}_+$, then $d^{\text{Cl}}(u_{\mathfrak{l}} \cdot u_{\bar{\mathfrak{l}}}) = \frac{1}{2}(\rho, \rho) \circ (u_{\mathfrak{l}} \cdot u_{\bar{\mathfrak{l}}})$.*

Proof: By Lemma 5.17, we have $d^{\text{Cl}}u_{\bar{\mathfrak{l}}} = \frac{1}{2}(\rho, -\rho) \circ u_{\bar{\mathfrak{l}}}$. The identity follows. \square

We have the following more general statement.

Proposition 5.19. *Let $\mathfrak{l}_{\pm} \subset \mathfrak{b}_{\pm}$ be a pair of Samelson algebras and $\rho_{\pm} \in \mathfrak{b}_{\pm}$ be the respective Weyl vectors. Let $u_+ \in \text{Cl}(\mathfrak{g}, \kappa)$ be a spinor defining $\bar{\mathfrak{l}}_- \boxplus \bar{\mathfrak{l}}_+$, then $d^{\text{Cl}}u_+ = \frac{1}{2}(-\rho_-, \rho_+) \circ u_+$.*

³In Lie theory, ρ is often used to denote the *Weyl vector* in \mathfrak{h}^* , i.e. half sum of positive roots. We use the same notation to denote the corresponding element in \mathfrak{h} via κ . In particular, $\rho \in i\mathfrak{t}$.

Proof: First suppose that $\mathfrak{b}_\pm = \mathfrak{b}$, then we have $\bar{\mathfrak{m}} = \bar{\mathfrak{l}}_+ \oplus \bar{\mathfrak{p}}$ with $\bar{\mathfrak{p}} \subseteq \mathfrak{h} \subseteq \mathfrak{b}$. Let $h \in \mathfrak{h}$,

$$d^{\text{Cl}}h = -\frac{1}{4} \sum_{\alpha \in R_+} ([h, a_\alpha] \cdot a_{-\alpha} + [h, a_{-\alpha}] \cdot a_\alpha) = -\frac{1}{4} \sum_{\alpha \in R_+} \alpha(h)(a_\alpha \cdot a_{-\alpha} - a_{-\alpha} \cdot a_\alpha)$$

Suppose that $h \in \mathfrak{h}$ and $h \notin \mathfrak{l}$. Write $\mathfrak{l} := \mathfrak{l}_+$, then

$$\begin{aligned} d^{\text{Cl}}(u_{\mathfrak{l}} \cdot h) &= d^{\text{Cl}}u_{\mathfrak{l}} \cdot h + (-1)^n u_{\mathfrak{l}} \cdot d^{\text{Cl}}h \\ &= \frac{1}{2} \rho \cdot u_{\mathfrak{l}} \cdot h + (-1)^n \frac{1}{2} u_{\mathfrak{l}} \cdot \rho \cdot h + (-1)^n u_{\mathfrak{l}} \cdot \left(-\frac{1}{4}\right) \sum_{\alpha \in R_+} \alpha(h)(2 - 2a_{-\alpha} \cdot a_\alpha) \\ &= \frac{1}{2} \rho \cdot u_{\mathfrak{l}} \cdot h + (-1)^{n+1} u_{\mathfrak{l}} \cdot \frac{1}{4} \sum_{\alpha \in R_+} (2\kappa(h_\alpha, h) - h_\alpha \cdot h) \\ &= \frac{1}{2} \rho \cdot u_{\mathfrak{l}} \cdot h + (-1)^{n+1} \frac{1}{2} u_{\mathfrak{l}} \cdot h \cdot \rho = \frac{1}{2} (-\rho, \rho) \circ (u_{\mathfrak{l}} \cdot h) \end{aligned}$$

The result then follows from mathematical induction on the dimension of \mathfrak{p} .

In general, let $\hat{g} \in D$ such that $\text{Ad}_{\hat{g}}(\mathfrak{b}_- \boxplus \mathfrak{b}_+) = \mathfrak{b} \boxplus \mathfrak{b}$, then $\hat{g} \circ u_+$ is a spinor defining $\text{Ad}_{\hat{g}}(\bar{\mathfrak{l}}_- \boxplus \bar{\mathfrak{l}}_+)$. From previous computation, we have

$$\hat{g} \circ d^{\text{Cl}}u_+ = d^{\text{Cl}}(\hat{g} \circ u_+) = \frac{1}{2} (-\rho, \rho) \circ (\hat{g} \circ u_+)$$

Thus

$$d^{\text{Cl}}u_+ = \frac{1}{2} \text{Ad}_{\hat{g}}^{-1}(-\rho, \rho) \circ u_+$$

and the statement follows. \square

Theorem 5.20. *Let $(K, \gamma; \mathbb{J}_\pm, \sigma)$ be a Lie algebraic generalized Kähler structure, and U_+ be the \mathbb{J}_+ -canonical line bundle. Then*

$$\deg_\pm(U_+) = -2|\rho|^2$$

Proof: The degrees are well defined according to Lemma 5.3. We prove it for induced generalized Kähler structures, and the general case is identical, with more cumbersome notations concerning the respective Weyl vectors. Furthermore, we consider only the $+$ -degree, and the $-$ -degree is similar. Let $\chi_+ = \mathcal{Q}(u_+)$, by Lemata 4.2 and 4.3, we have

$$d_\gamma \chi_+ = \frac{1}{2} \mathfrak{X}_{(-\rho, \rho)} \cdot \chi_+$$

Let D_{U_+} denote the structure of \mathbb{J}_+ -holomorphic line bundle on $U_+ = \langle \chi_+ \rangle$. For $s = \mathfrak{X}_{(a, a')}$ with $a \in \bar{\mathfrak{l}}_-$ and $a' \in \bar{\mathfrak{l}}_+$, we have

$$D_{U_+, s} \chi_+ = \mathfrak{X}_{(a, a')} \cdot \mathfrak{X}_{(-\rho, \rho)} \cdot \chi_+ = \kappa(a' + a, \rho) \chi_+$$

Let $\rho_\pm^{1,0} \in \mathfrak{l}_\pm$ denote the projections of ρ to \mathfrak{l}_\pm respectively, and

$$(5.5) \quad \varphi_+ = \kappa(\theta^r, \rho_+^{1,0}) \text{ and } \varphi_- = -\kappa(\theta^l, \rho_-^{1,0})$$

Then, the induced structures of I_\pm -holomorphic line bundles are

$$(5.6) \quad D_+ \chi_+ = \varphi_+ \otimes \chi_+ \text{ and } D_- \chi_+ = \varphi_- \otimes \chi_+$$

A natural Hermitian metric on U_+ is defined by $|\chi_+|^2 = 1$. The Chern connection for D_+ is

$${}^C \nabla^+ \chi_+ = (\varphi_+ - \bar{\varphi}_+) \otimes \chi_+$$

whose curvature is given by

$$F_+ = d(\varphi_+ - \bar{\varphi}_+)$$

Use the basis $\{b_j\}_{j=1}^n$ of $\mathfrak{l} := \mathfrak{l}_+$ and the dual basis $\{\bar{b}_j\}$ of $\bar{\mathfrak{l}}$. Then $\{b_j, \bar{b}_j\}$ gives a basis of \mathfrak{g} . Let $\{b_j^*, \bar{b}_j^*\}$ be the dual basis of \mathfrak{g}^* , e.g. $\langle b_i^*, b_j \rangle = \delta_{ij}$. Then the value of ω_+ at $e \in K$ given by

$$\omega_+(e) = \frac{i}{2} \sum_i b_i^* \wedge \bar{b}_i^*$$

To compute the degree $\deg_+(U_+)$, we are interested in $F_+ \wedge \omega_+^{n-1}$. Since all objects involved are right invariant, we only have to carry out the computation at $e \in K$, i.e. on the level of Lie algebra. The $b_i^* \wedge \bar{b}_i^*$ terms of $d\varphi_+(e)$ are

$$\begin{aligned} d\kappa(\theta^r, \rho_+^{1,0}) &= -\kappa([\theta^r, \theta^r], \rho_+^{1,0}) \\ \implies -\sum_{i,j} \kappa([b_i, \bar{b}_i], \rho_+^{1,0}) b_i^* \wedge \bar{b}_i^* &= -\sum_{\alpha \in R_+} \kappa(h_\alpha, \rho_+^{1,0}) b_\alpha^* \wedge \bar{b}_\alpha^* \\ &= -\sum_{\alpha \in R_+} \kappa(h_{\alpha,+}^{0,1}, \rho_+^{1,0}) b_\alpha^* \wedge \bar{b}_\alpha^* \end{aligned}$$

It follows that the $b_i^* \wedge \bar{b}_i^*$ terms of $F_+(e)$ are

$$-\sum_{\alpha \in R_+} \left(\kappa(h_{\alpha,+}^{0,1}, \rho_+^{1,0}) + \overline{\kappa(h_{\alpha,+}^{0,1}, \rho_+^{1,0})} \right) b_\alpha^* \wedge \bar{b}_\alpha^* = -2 \sum_{\alpha \in R_+} \Re(\kappa(h_{\alpha,+}^{0,1}, \rho_+^{1,0})) b_\alpha^* \wedge \bar{b}_\alpha^*$$

and

$$F_+(e) \wedge \omega_+^{n-1}(e) = -\frac{2n!i^{n-1}}{2^{n-1}} \sum_{\alpha \in R_+} \Re(\kappa(h_{\alpha,+}^{0,1}, \rho_+^{1,0})) \wedge_i (b_i^* \wedge \bar{b}_i^*) = -\frac{4n!i^{n-1}}{2^{n-1}} \Re(\kappa(\rho_+^{0,1}, \rho_+^{1,0})) \wedge_i (b_i^* \wedge \bar{b}_i^*)$$

Since $\kappa(\rho, \rho) = 2\Re(\kappa(\rho_+^{0,1}, \rho_+^{1,0}))$ we have

$$\deg_+(U_+) = \frac{i}{2} \frac{\int_K F_+ \wedge \omega_+^{n-1}}{\int_K \omega_+^n} = \frac{i}{2} \frac{-\frac{2n!i^{n-1}}{2^{n-1}} \kappa(\rho, \rho) \wedge_i (b_i^* \wedge \bar{b}_i^*)}{\frac{n!i^n}{2^n} \wedge_i (b_i^* \wedge \bar{b}_i^*)} = -2\kappa(\rho, \rho)$$

□

Remark 5.21. Although Proposition 5.19 and Theorem 5.20 concern only the structure \mathbb{J}_+ , it's easy to see that similar results hold for \mathbb{J}_- . Namely, we have

$$\deg_\pm(U_-) = -2|\rho|^2$$

where the \deg_\pm in this case are respect to $\pm I_\pm$. These results may be interpreted as a certain *positivity* of the Lie algebraic generalized Kähler structures.

We recover the following result from [7].

Corollary 5.22. *Let $(K, \gamma; \mathbb{J}_\pm, \sigma)$ be a Lie algebraic generalized Kähler structure. Then \mathbb{J}_+ is generalized Calabi-Yau iff K is abelian, i.e. a torus.*

Proof: Since $\rho = 0 \iff K$ is a torus, we see that the \mathbb{J}_+ -canonical bundle is holomorphically trivial iff K is a torus. □

Corollary 5.23. *Let T be an even dimensional torus, then any invariant generalized Kähler structure on T is a generalized Calabi-Yau metric structure.*

Proof: For a torus, $\rho = 0$ and χ_+ is a non-vanishing holomorphic section of U_+ , which implies that \mathbb{J}_+ is generalized Calabi-Yau. That \mathbb{J}_- is also generalized Calabi-Yau in this case follows from Remark 5.21. Proposition 4.2 d) in [1] gives the comparison of lengths of the spinors. □

5.6. Hodge decomposition. The Hodge decomposition of the twisted de Rham complex $(\Omega^*(K), d_\gamma)$ induces the *Hodge decomposition* of $\text{Cl}(\mathfrak{g}, \kappa)$. Consider the spinor actions of $\hat{J}_+ := J_- \oplus J_+$ and $\hat{J}_- := -J_- \oplus J_+$ on $\text{Cl}(\mathfrak{g}, \kappa)$.

Lemma 5.24. *Let $\{b_{j,\pm}\}$ be a complex basis of \mathfrak{l}_\pm and $\{\bar{b}_{j,\pm}\}$ the dual basis of $\bar{\mathfrak{l}}_\pm$ with respect to κ , i.e. $\kappa(b_{i,\pm}, \bar{b}_{j,\pm}) = \delta_{ij}$. Define $\tau_{J_\pm} \in \text{Cl}(\mathfrak{g}, \kappa)$ by*

$$\tau_{J_+} = -\frac{ni}{2} + \frac{i}{2} \sum_{j=1}^n b_{j,+} \cdot \bar{b}_{j,+} \text{ and } \tau_{J_-} = \frac{ni}{2} - \frac{i}{2} \sum_{j=1}^n \bar{b}_{j,-} \cdot b_{j,-}$$

Let $\hat{\tau}_\pm = (\pm\tau_{J_-}, \tau_{J_+})$ then the action of \hat{J}_\pm on $\text{Cl}(\mathfrak{g}, \kappa)$ is given by

$$u \mapsto \hat{\tau}_\pm \circ u = \tau_{J_+} \cdot u \mp u \cdot \tau_{J_-}$$

Proof: Since κ is Hermitian with respect to both J_\pm , τ_{J_\pm} are the corresponding elements in $\text{Cl}(\mathfrak{g}, \kappa)$ representing J_\pm :

$$\tau_{J_\pm} = \frac{1}{4} \sum_{j=1}^n (J_\pm(b_{j,\pm}) \cdot \bar{b}_{j,\pm} + J_\pm(\bar{b}_{j,\pm}) \cdot b_{j,\pm})$$

The statements follow from straightforward computations and the definition of spinor actions. \square

Corollary 5.25. *Let u_+ be the pure spinor defining $\bar{\mathfrak{l}}_- \boxplus \bar{\mathfrak{l}}_+$, then $\hat{\tau}_+ \circ u_+ = -inu_+$ and $\hat{\tau}_- \circ u_+ = 0$.*

Proof: It follows from (5.4) that $\tau_{J_+} \cdot u_+ = -\frac{ni}{2}u_+$ and $u_+ \cdot \tau_{J_-} = \frac{ni}{2}u_+$. \square

Since \mathfrak{l}_\pm are isotropic with respect to κ , we have $\wedge^* \mathfrak{l}_\pm \subset \text{Cl}(\mathfrak{g}, \kappa)$.

Proposition 5.26. $\text{Cl}(\mathfrak{g}, \kappa) = \bigoplus_{r,s} \mathfrak{U}_{r,s}$, where $\mathfrak{U}_{r,s} := \wedge^r \mathfrak{l}_+ \cdot u_+ \cdot \wedge^s \mathfrak{l}_-$ with $r = p+q-n$ and $s = p-q$. Moreover, $\mathfrak{U}_{r,s}$ is the $i(r,s)$ -eigensubspace of the spinor action by $(\hat{\tau}_+, \hat{\tau}_-)$.

Proof: First we note that $\text{Cl}(\mathfrak{g}, \kappa) = \text{Span}\{\mathfrak{U}_{r,s}\}$. Then the second statement implies the first one. For any $p, q = 0, 1, \dots, n$, we compute the action of $\hat{\tau}_+$ on

$$u := v_+ \cdot u_+ \cdot v_- := b_{1,+} \cdot \dots \cdot b_{p,+} \cdot u_+ \cdot b_{1,-} \cdot \dots \cdot b_{q,-}$$

and show that it is an ir -eigenvector, with $r = p+q-n$. The cases where v_\pm are general basis elements of $\wedge^* \mathfrak{l}_\pm$ are similar.

$$\begin{aligned} \hat{\tau}_+ \circ u &= \tau_{J_+} \cdot u - u \cdot \tau_{J_-} = -inu + \frac{i}{2} \sum_{j=1}^n (b_{j,+} \cdot \bar{b}_{j,+} \cdot u + u \cdot \bar{b}_{j,-} \cdot b_{j,-}) \\ &= -inu + \frac{i}{2} \sum_{j=1}^n (b_{j,+} \cdot \bar{b}_{j,+} \cdot v_+ \cdot u_+ \cdot v_- + v_+ \cdot u_+ \cdot v_- \cdot \bar{b}_{j,-} \cdot b_{j,-}) \\ &= -inu + \frac{i}{2} \sum_{j=1}^p (2v_+ \cdot u_+ \cdot v_-) + \frac{i}{2} \sum_{j=1}^q (2v_+ \cdot u_+ \cdot v_-) \\ &= i(p+q-n)u = iru \end{aligned}$$

The computation for the action of $\hat{\tau}_-$ is similar. \square

5.7. Cohomology of $\bar{\mathcal{L}}$. From now on, we write $\mathbb{J} := \mathbb{J}_+$, $L := L_+$.

Let $\{\xi_i^\pm : i = 1, \dots, n\}$ be a basis of the dual \mathfrak{l}_\pm^* of the complex Lie subalgebra $\bar{\mathfrak{l}}_\pm \subset \mathfrak{g}$, respecting the decompositions $\bar{\mathfrak{l}}_\pm = \mathfrak{t}_{\pm,(0,1)} \oplus \mathfrak{r}_\pm$. For $i = 1, \dots, n$, let α_i (respectively, β_i) be the right invariant (respectively, left invariant) 1-form on K such that $\alpha_i(e) = \xi_i^+$ (respectively, $\beta_i(e) = \xi_i^-$). Then $\{\alpha_i\}_{i=1}^n$ is the basis of right invariant $I_+-(0,1)$ -forms and $\{\beta_i\}_{i=1}^n$ is the basis of left invariant $I_--(0,1)$ -forms. We use the same notations, α_i and β_i , to denote the corresponding elements in $\Omega^1(\bar{\mathcal{L}}_\pm)$. For $P \subset \{1, 2, \dots, n\}$, we use ξ_P^+ and β_P to denote $\bigwedge_{i \in P} \xi_i^+$ and $\bigwedge_{i \in P} \beta_i$, and etc.

Proposition 5.27. \mathcal{A}_\pm are trivial as I_\pm -holomorphic vector bundles.

Proof: By definition, $\{\alpha_i\}$ and $\{\beta_i\}$ are global frames of $\bar{\ell}_+^*$ and $\bar{\ell}_-^*$ respectively. By Lemma 5.4, $\bar{\partial}_-\alpha_i = 0$ and $\bar{\partial}_+\beta_i = 0$ for all i . Lemma 3.5 implies that they are global holomorphic frames of \mathcal{A}_\pm^* respectively. Thus \mathcal{A}_\pm are holomorphically trivial. \square

Corollary 5.28. The first page of the first spectral sequence associated to $\Omega^{p,q}(\bar{L})$ is

$${}_IE_1^{p,q} = \left(\wedge^p \bar{\ell}_+^* \otimes \wedge^q \mathfrak{t}_-^{0,1}, d_{\bar{\ell}_+} \otimes id \right)$$

where $\mathfrak{t}_-^{0,1} = (\mathfrak{t}_{-(0,1)})^*$.

Proof: The description of the first page of the spectral sequence in §2.2 gives

$${}_IE_1^{p,q} = H_{\bar{\partial}_-}^{0,q}(\wedge^q \mathcal{A}_-^*)$$

By Proposition 5.27, $\wedge^q \mathcal{A}_-$ is trivial as well. The explicit trivialization gives

$${}_IE_1^{p,q} = H_{\bar{\partial}_-}^{0,q}(\wedge^q \mathcal{A}_-^*) = \wedge^p \bar{\ell}_+^* \otimes H_{\bar{\partial}_-}^{0,q}(K)$$

The expression of the E_1 -terms then follows from the general fact (cf. [28]) that

$$H_{\bar{\partial}_-}^{0,q}(K) \cong \wedge^q \mathfrak{t}_-^{0,1}$$

To compute the induced differential d_1 , we note that $\bar{\partial}_+$ coincides with the differential $d_{\bar{\ell}_+}$ on $\wedge^p \bar{\ell}_+^*$. By Alexandrov-Ivanov [3], $H_{\bar{\partial}_-}^{0,q}(K)$ can be represented by $\bar{\partial}_-$ -Harmonic forms, which are left invariant. From Corollary 5.5, $\bar{\partial}_- = 0$ on $\wedge^q \mathfrak{t}_-^{0,1}$. Thus, $\bar{\partial}_+$ induces $d_1 = d_{\bar{\ell}_+} \otimes id$. \square

Proposition 5.29. The first spectral sequence associated to $\Omega^{p,q}(\bar{L})$ collapses at E_2 , where

$${}_IE_2^{p,q} = \wedge^p \mathfrak{t}_+^{0,1} \otimes \wedge^q \mathfrak{t}_-^{0,1}$$

Proof: Either using the Hochschild-Serre spectral sequence (Hochschild-Serre [18]) or quoting [28] again, we have $H^*(\bar{L}_+) \cong H^*(\mathfrak{t}_+^{0,1}) = \wedge^* \mathfrak{t}_+^{0,1}$. To obtain the differential d_2 , we note that a basis of ${}_IE_2^{p,q}$ is given by the classes represented by the forms $\alpha_Q \wedge \beta_{Q'}$, where $|Q| = p$, $|Q'| = q$, such that $\xi_Q^+ \in \wedge^p \mathfrak{t}_+^{0,1}$ and $\xi_{Q'}^- \in \wedge^q \mathfrak{t}_-^{0,1}$. Since

$$\bar{\partial}_+(\alpha_Q \wedge \beta_{Q'}) = \bar{\partial}_+ \alpha_Q \wedge \beta_{Q'} + (-1)^p \alpha_Q \wedge \bar{\partial}_+ \beta_{Q'} = 0$$

we see that $d_2 = 0$ on ${}_IE_2^{p,q}$. \square

From Proposition 3.6, we obtain:

Corollary 5.30. $\mathbb{H}^*(\mathcal{A}_+) \cong \mathbb{H}^*(\mathcal{A}_-) \cong H^*(\bar{L}) \cong \wedge^*(\mathfrak{t}_-^{0,1} \oplus \mathfrak{t}_+^{0,1})$. \square

5.8. \mathbb{J} -Picard group. Let $V = K \times \mathbb{C}$ be the topologically trivial complex line bundle on K . The I_+ -holomorphic structures on V is then classified by the classical Picard group with respect to I_+

$$\text{Pic}_0^+(K) = \frac{H^1(K; \mathcal{O}_+)}{H^1(K; \mathbb{Z})}$$

By [28], $H^1(K; \mathcal{O}_+) \cong \mathbb{C}^r$ where $2r$ is the rank of K . In general, $H^1(K; \mathbb{Z})$ may not be a full lattice in $H^1(K; \mathcal{O}_+)$. For example, when K is semi-simple and simply connected, $H^1(K; \mathbb{Z}) = \{0\}$ and $\text{Pic}_0^+(K) \cong \mathbb{C}^r$. In general, when $H^1(K; \mathbb{Z})$ has no torsion, the Picard group is of the form

$$\text{Pic}_0^+(K) \cong \mathbb{C}^{r_1} \times (\mathbb{C}^\times)^{r_2} \times (S^1)^{2r_3}, \text{ with } r_1 + r_2 + r_3 = r$$

For example, let $T = (S^1)^2$ be the 2-torus, $H = SU(2) \times S^1$ the Hopf surface, then we have

$$\text{Pic}_0(T) \cong (S^1)^2, \text{ Pic}_0(H) \cong \mathbb{C}^\times \text{ and } \text{Pic}_0(SU(3)) \cong \mathbb{C}$$

Let $\mathbb{P}\mathrm{ic}_0(K)$ denote the space classifying \mathbb{J} -holomorphic structures on V . Then $\mathbb{P}\mathrm{ic}_0(K)$ is a group under the tensor product, with identity represented by the trivial connection

$$D_0 : C^\infty(V) \rightarrow C^\infty(V \otimes \overline{L}^*) : D_0(u) = 0$$

where u is the constant section with value 1. We note that in this case, $C^\infty(V \otimes \overline{L}^*) \cong \Omega^1(\overline{L})$ and D_0 coincides with $d_{\overline{L}}$. Any \overline{L} -connection D on V is of the form

$$D = d_{\overline{L}} + \omega, \text{ with } \omega \in \Omega^1(\mathrm{End}(V); \overline{L}) \cong \Omega^1(\overline{L})$$

The flatness of D is equivalent to the Maurier-Cartan equation

$$d_{\overline{L}}\omega + \omega \wedge \omega = d_{\overline{L}}\omega = 0$$

It then follows that

$$\mathbb{P}\mathrm{ic}_0(K) \cong \frac{\ker d_{\overline{L}}}{\mathcal{G}(V)}$$

where $\mathcal{G}(V) = C^\infty(\mathrm{Aut}(V))$ is the gauge group acting on the space of \overline{L} -connections:

$$g \circ D := D + g^{-1}d_{\overline{L}}g$$

Infinitesimally, the tangent space of $\mathbb{P}\mathrm{ic}_0(K)$ at D_0 is given by $H^1(\overline{L}) \cong \mathfrak{t}_+^{0,1} \oplus \mathfrak{t}_-^{0,1} \cong \mathbb{C}^{2r}$.

On any generalized Kähler manifold, Lemma 2.3, together with the isomorphism of categories in Proposition 3.15, gives the exact sequence of complex analytic group homomorphisms:

$$0 \rightarrow \mathrm{Pic}_0^{A+} \rightarrow \mathbb{P}\mathrm{ic}_0 \rightarrow \mathrm{Pic}_0^+$$

On K , the sequence above can be completed to a split short exact sequence.

Proposition 5.31. $\mathbb{P}\mathrm{ic}_0(K) \cong \mathrm{Pic}_0^+(K) \times \mathrm{Pic}_0^{A+}(K) \cong \mathrm{Pic}_0^+(K) \times \mathbb{C}^r$.

Proof: We only have to show that $\mathrm{Pic}_0^{A+}(K) \cong \mathbb{C}^r$, then dimension counting implies that the forgetful map $\mathbb{P}\mathrm{ic}_0(K) \rightarrow \mathrm{Pic}_0^+(K)$ is a submersion. The result then follows.

On the trivial I_+ -holomorphic bundle \mathcal{O}_+ , let $\partial_{\mathcal{A}_+,0}$ be the trivial \mathcal{A}_+ -connection. Then any flat \mathcal{A}_+ -connection on \mathcal{O}_+ is of the form

$$\partial_{\mathcal{A}_+} = \partial_{\mathcal{A}_+,0} + \alpha, \text{ with } \alpha \in \mathcal{A}_+^* \text{ and } \partial_{\mathcal{A}_+,0}\alpha = 0$$

We note that the only holomorphic bundle automorphisms on \mathcal{O}_+ are multiplications by elements of \mathbb{C}^\times . Since \mathcal{A}_+ is a trivial I_+ -holomorphic bundle by Proposition 5.27, similar arguments as in the proof of Proposition 5.29 gives $\mathrm{Pic}_0^{A+}(K) \cong \mathbb{C}^r$. \square

Remark 5.32. The \mathbb{J} -Picard group on a generalized Kähler manifold can also be seen from the bi-Hermitian interpretation as following. By Lemma 3.10, V is a \mathbb{J} -holomorphic line bundle iff it is I_\pm -holomorphic and the induced differentials D_\pm satisfy the commutation relation. For $V = M \times \mathbb{C}$, by an abuse of notations, let D_0 denote both of the trivial I_\pm -holomorphic structures on V , then an I_\pm -holomorphic structure is given by a pair

$$(\alpha_+, \alpha_-) \in \Omega_+^1(M) \times \Omega_-^1(M) \text{ such that } \overline{\partial}_+\alpha_+ = \overline{\partial}_-\alpha_- = 0$$

where the complex gauge group \mathcal{G} acts diagonally. The commutation relation in this case becomes

$$\overline{\partial}_-\alpha_+ + \overline{\partial}_+\alpha_- = 0$$

Thus, the \mathbb{J} -Picard group can also be seen as the quotient

$$\mathbb{P}\mathrm{ic}_0(M) = \frac{\{(\alpha_+, \alpha_-) : \overline{\partial}_-\alpha_+ + \overline{\partial}_+\alpha_- = 0\}}{\mathcal{G}}$$

This is in general different from $\mathrm{Pic}_0^+(M) \times \mathrm{Pic}_0^-(M)$.

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